Solve the following Exercises.

Exercise 1 (10 points 3-3-4): Let $K = F(a)$ be a finite extension of $F$. For $\alpha \in K$, define $\phi_\alpha : K \to K$ by $\phi_\alpha(x) = x\alpha$.
(1) Prove that $\phi_\alpha$ is an $F$-linear transformation.
(2) Prove that $det(xI - \phi_\alpha)$ is the minimal polynomial of $\alpha$, where $I$ is the identity $F$-transformation.
(3) For which $\alpha \in K$ is $det(xI - \phi_\alpha)$ the minimal polynomial of $\alpha$ over $F$. 
**Exercise 2** (10 points): Let $K|F$ be a field extension. Prove that $K$ is algebraic over $F$ if and only if every monic polynomial of $K[X]$ is a divisor of some non-zero polynomial in $F[X]$. 
Exercise 3 (10 points 3-4-3): Let $a$ be an odd integer.
(1) Prove that $X^3 + 4X - a^2$ is irreducible in $\mathbb{Q}[X]$.
(2) Let $\mathbb{E}$ be a splitting field of $X^4 - aX - 1$, $\theta$ a root of $X^4 - aX - 1$ in $\mathbb{E}$ and $K = \mathbb{Q}(\theta)$. Prove that for every $\alpha, \beta$ in $\mathbb{E}$, if $X^2 + \alpha + \beta$ divides $X^4 - aX - 1$, then the minimal polynomial $P_{\alpha^2, \mathbb{Q}}$ of $\alpha^2$ over $\mathbb{Q}$ is $X^3 + 4X - a^2$.
(3) Find $[K : \mathbb{Q}]$. 

Exercise 4 (10 points 3-3-4): Let $F$ be a field of positive characteristic $p$, $K = F(X, Y)$ and $L = F(X^p, Y^p)$.

(1) Find $[K : L]$.

(2) Prove that $u^p \in L$ for every $u \in K$.

(3) Prove that the extension $K|_L$ has infinitely many intermediate fields.
Exercise 5 (10 points): Let $K$ be a field and $\sigma \in Aut(K)$ has infinite order. Let $F$ be the fixed field of $\sigma$. Prove that if $K$ is algebraic over $F$, then $K$ is a normal extension of $F$. 
Exercise 6 (10 points 5-5): Let $K$ and $L$ be two normal extensions of a field $F$ and suppose that $K$ and $L$ are subfields of a common field $E$.

1) Prove that $K \cap L$ is a normal extension of $F$.
2) Prove that $F(K \cup L)$ is a normal extension of $F$. 


Exercise 7 (10 points 3-4-3):
(1) Prove that for any root $\theta$ of the polynomial $X^4 - 2$ of $\mathbb{Q}[X]$, $\mathbb{Q}(\theta)$ is not a normal extension of $\mathbb{Q}$.
(2) Find three extensions $K_1 \subsetneq K_2 \subsetneq K_3$ such that $K_2$ is normal over $K_1$, $K_1$ is normal over $\mathbb{Q}$ but $K_2$ is not normal over $\mathbb{Q}$.
(3) $K_3$ normal over $\mathbb{Q}$. 
Exercise 8 (10 points 3-4-3): Let $K|F$ be an extension of fields of characteristic $p \neq 0$ and set $L = \{x \in K | x^{p^r} \in F \text{ for some integer } r \geq 0\}$.

(1) Prove that $L$ is an intermediate field (i.e., $F \subseteq L \subseteq K$).

(2) Prove that if $K$ is perfect, then $L$ is perfect.

(3) Prove that every $F$-automorphism of $K$ is an $L$-automorphism.
Exercise 9 (10 points): Let $F \subset L \subset K$ be extension of fields such that $K|_L$ is normal and $L|_F$ is purely inseparable. Prove that $K|_F$ is normal.
Exercise 10 (10 points 3-4-3): Let $F$ be a field of characteristic $p \neq 0$, $n$ a positive integer and $a \in F$. 

(1) Prove that $X^{p^n}$ has only one irreducible divisor $f$ in $F[X]$. 

(2) Prove that there exists a positive integer $m \leq n$ such that $X^{p^n} - a = f^{p^m}$. 

(3) Prove that there exists $b \in F$ such that $f = X^{p^{n-m}} - b$. 