Exercise 1 (10 points 5-5): Let $K$ be an extension of $F$, and $[F, K]$ the set of all intermediate fields between $F$ and $K$.

(1) Prove that if $K = F(a, b)$ for some $a, b \in K$ and $[F, K]$ is finite, then $K$ is a simple extension of $F$.

(2) Prove that if $[F, K]$ is finite, then $K$ is a simple extension of $F$. (Hint, use Question 1).
Exercise 2 (10 points 5-5): Let $F$ be a finite field of order $q$ and let $f \in F[X]$ an irreducible polynomial over $F$.

1-Prove that, for any positive integer $n$, $f$ divides $X^q^n - X$ if and only if the degree of $f$ divides $n$.

2-Prove that, for any positive integer $n$, $F[X]$ contains an irreducible polynomial of degree $n$. 
Exercise 3 (10 points 5-5): Let $K$ be a finite field of order 4.
(1) Prove that $K = \mathbb{F}_2(\alpha)$ where $\alpha^2 + \alpha + 1 = 0$.
(2) Find the irreducible factorization of $X^4 + 1$ over $\mathbb{F}_3$. 
Exercise 4 (10 points 3-3-4): (1) Let $K$ be a purely inseparable extension of $F$. Prove that $Tr(a) = 0$ for any $a \in K$.

(2) Let $K|F$ be an extension of finite fields. Prove that the norm map $N_{K|F}$ is surjective.

(3) Let $p$ be an odd prime, $\omega$ a primitive $p$th root of unity and $K = \mathbb{Q}(\omega)$. Prove that $Nr_{K|\mathbb{Q}}(1 - \omega) = p$. 
Exercise 5 (10 points 3-3-4): Let $F$ be a field.
(1) Prove that if $a \in F$, then the splitting field of $X^n - a$ over $F$ is $F(b, \omega)$ where $b^n = a$ and $\omega$ is a primitive $n$th root of unity.
(2) Set $N = F(b, \omega)$, $K = F(b)$ and $L = F(\omega)$. Prove that the extension $L|_F$ is Galois and the extension $N|_L$ is cyclic.
(3) Suppose that the minimal polynomial of $\omega$ over $F$ is $P_{F,\omega}(X) = (X - \omega)(X - \omega^{-1})$ and that $[N : L] = n$ Prove that there exits $\sigma \in Gal(N|_F)$ of order $n$ such that $\sigma(b) = b\omega$ and $\sigma(\omega) = \omega$. 