Name:	ID:

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Final Exam for Math 572, Semester 121

Note: Throughout the exam, C is a generic positive constant.

Problem 1. Consider the following two-point BVP:

$$\begin{cases} -(a(x)u')' = 1 \quad \text{in } (0,1) \\ u'(0) = 0 \quad \text{and} \quad u(1) = 0, \end{cases} \tag{1}$$

where $a(x) \geq a_0 > 0$ for all $0 \leq x \leq 1$.

a) Define a weak form of the above problem.

b) Define a piecewise-linear finite-element method using arbitrary mesh

$$0 = x_0 < x_1 < \cdots < x_M = 1.$$

c) Write the obtained numerical scheme in a matrix form. (Do not evaluate the elements of the matrices).

Problem 2. Consider the following fourth-order BVP:

$$\begin{cases} u^{(4)} + c\,u = f(x) \quad \text{in } (0,1) \\ u(0) = u'(0) = u(1) = u'(1) = 0, \end{cases} \tag{2}$$

where c is a positive constant and f is a smooth function.

a) Show that this problem may be formulated as: Find $u \in W$ such that

$$(u'', v'') + c(u, v) = (f, v) \quad \text{for all} \quad v \in W$$

where

$$W = \{v \in H^2(0,1) : v(0) = v(1) = v'(0) = v'(1) = 0\}$$

and $(\cdot,\cdot)$ is the L_2-inner product.

b) Define a piecewise-cubic FEM using equidistant mesh each is of length h.

c) Show that the finite element solution u_h (introduced in part (b)) exists and is unique.

d) Assume that $\inf_{\chi \in S_h} \|u'' - \chi''\| \leq Ch^2 \|u^{(4)}\|$. Prove that $\|u'' - u_h''\| \leq Ch^2 \|u^{(4)}\|$.

Problem 3. Consider the following one dimensional parabolic, initial BVP:

$$\begin{cases} u_t - 3\,u_{xx} = f \quad \text{in } (x,t) \in (0,1) \times (0,1) \\ u(0,t) = u(1,t) = 0 \quad \text{for } t \in (0,1) \\ u(x,0) = v(x) \quad \text{for } x \in (0,1). \end{cases} \tag{3}$$

Here $u = u(x,t)$ and $f = f(x,t)$.

Choosing N and M to be positive integers, we define the grid points:

$$x_n = nh \ \text{ for } \ 0 \leq n \leq N, \ \text{ where } h = 1/N$$
$$t_m = mk \ \text{ for } \ 0 \leq m \leq M, \ \text{ where } k = 1/M,$$

and introduce the grid functions

$$U_n^m \approx u(x_n,t_m), \quad v_n = v(x_n) \text{ and } f_n^m = f(x_n,t_m).$$

Consider the finite difference (FD) scheme:

$$\frac{3U_n^m - 4U_n^{m-1} + U_n^{m-2}}{2k} - 3\frac{U_{n+1}^m - 2U_n^m + U_{n-1}^m}{h^2} = f_n^m \text{ for } 1 \le n \le N-1 \text{ and } 2 \le m \le M$$

$$\frac{U_n^1 - U_n^0}{k} - 3\frac{U_{n+1}^1 - 2U_n^1 + U_{n-1}^1}{h^2} = f_n^1 \text{ for } 1 \le n \le N-1$$

$$U_n^0 = v_n \text{ for } 1 \le n \le N-1$$

$$U_0^m = U_N^m = 0 \text{ for } 0 \le m \le M.$$

a) Use Taylor series expansion to show that

$$\frac{3g(t) - 4g(t-k) + g(t-2k)}{2k} = g'(t) - \frac{1}{3}g'''(t)k^2 + O(k^3)$$

and

$$\frac{g(x+h) - 2g(x) + g(x-h)}{h^2} = g''(x) + \frac{1}{12}g^{(4)}(x)h^2 + O(h^4).$$

b) Assume that the given FD scheme is stable. What is the expected accuracy (in time and space) of the approximation $U_n^m \approx u(x_m,t_n)$.
c) Write the given FD scheme in the matrix form as follows:

$$A\mathbf{U}^1 = V + kF^1$$

$$B\mathbf{U}^m = 4\mathbf{U}^{m-1} - \mathbf{U}^{m-2} + 2kF^m \text{ for } 2 \le m \le M$$

where $\mathbf{U}^m = [U_1^m, U_2^m, \cdots, U_{N-1}^m]^T$, $F^m = [f_1^m, f_2^m, \cdots, f_{N-1}^m]^T$, $V = [v_1, v_2, \cdots, v_{N-1}]^T$, and A and B are $(N-1)\times(N-1)$ tridiagonal matrices.

Problem 4. Consider the following problem:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(u) & \text{for } (x,t) \in \Omega \times (0,T] \\ u(0,t) = u(1,t) = 0 & \text{for } t \in (0,T) \\ u(x,0) = v(x) & \text{for } x \in \Omega \end{cases} \tag{4}$$

where $\Omega = (0,1)$. Assume that $|f(s) - f(q)| \le C|s-q|$ for any $s, q \in \mathbb{R}$ and $f(0) = 0$.

a) Show that $\|u(t)\| \le C\|v\|$ for any $t \in (0,T]$.

Hint You may need to use the continuous Gronwall's inequality, that is, if

$$g(t) \le \ell(t) + \int_0^t g(s)\,ds \quad \text{for any } t \in (0,T]$$

where the functions g and ℓ are non-negative and ℓ is non-decreasing, then $g(t) \le C\ell(t)$ for $t \in (0,T]$.

b) Discretize problem (4) in time by using backward Euler scheme over a uniform mesh consists of N subintervals and each is of length k.

c) Say that $U^n \approx u(t_n)$ is the backward Euler solution, for $n = 1,\cdots,N$, and assume that the time-step size k is sufficiently small. Prove the following stability property:

$$\|U^n\| \le C\|v\| \quad \text{for } n = 1,\cdots,N.$$

Hint You may need to use the discrete version of Gronwall's inequality, that is, if

$$g(t_n) \le \ell(t_n) + Ck\sum_{j=1}^{n-1} g(t_j) \quad \text{for } n = 1,\cdots,N$$

where the functions g and ℓ are non-negative and ℓ is non-decreasing, then $g(t_n) \le C\ell(t_n)$ for $n = 1,\cdots,N$.

d) Assume that u is sufficiently regular. Prove that $\|U^n - u(t_n)\| \le Ck$.