Solve 5 problems (out of 7) from the below list.

(1) Let $P = (p)$ be a principal prime ideal of a ring $R$ and $J = \bigcap P^n$. Prove the following:
   (a) $Q$ is a prime ideal of $R$ strictly contained in $P \implies Q \subset J$.
   (b) $p \notin Z(R) \implies J = pJ$.
   (c) $p \notin Z(R) \implies J \in \text{Spec}(R)$.
   (d) $R$ is an integral domain and $J$ is finitely generated $\implies J = 0$ and $ht(P) = 1$.
   (e) $J$ is finitely generated $\implies ht(P) \leq 1$.

(2) Let $R$ be a ring and $u \in U(R)$. Prove that $R[u] \cap R[u^{-1}]$ is integral over $R$.

(3) (The homogeneous Nullstellensatz) Let $K$ be a field and $R = K[x_1, \ldots, x_n]$. An ideal $I$ of $R$ is homogeneous if $f \in I \implies$ all the homogenous constituents of $f \in I$. A variety $V (\subseteq K^n)$ is a cone if $(a_1, \ldots, a_n) \in V \implies (ta_1, \ldots, tan) \in V$ for all $t \in K$.
   (a) Prove that if $I$ is homogeneous, then $V(I)$ is a cone; where $V(I)$ denotes the variety consisting of all points of $K^n$ where all polynomials in $I$ vanish.
   (b) Prove that if $V$ is a cone and $K$ is infinite, then $J(V)$ is homogeneous; where $J(V)$ denotes the ideal of polynomials vanishing on $V$.
   (c) Prove that if $K$ is algebraically closed, then the radical of a homogeneous ideal is homogeneous.

(4) Let $R$ be an integral domain with quotient field $K$. Prove the following:
   (a) $S^{-1}R$ integral over $R \implies R = S^{-1}R$ (for any given multiplicative subset $S$ of $R$).
   (b) Every ring between $R$ and $K$ is integrally closed $\implies R$ is Prüfer.
   (c) Every ring between $R$ and $K$ is a localization $\implies R$ is Prüfer.

(5) Let $R$ be an integral domain, $K$ its quotient field, and $R'$ its integral closure. Let $T$ be a ring between $R$ and $K$, and $D$ the conductor of $T$ relative to $R$, i.e., $D = (R :_RT)$. Prove:
   (a) $R$ is Noetherian and $D \neq 0 \implies T$ is a finitely generated $R$-module
   (b) $D \not\subset P \in \text{Spec}(R)$ and $Q \in \text{Spec}(T)$ with $Q \cap R = P \implies R_P = T_Q$.
   (c) $T$ is a finitely generated $R$-module and $P \in \text{Spec}(R)$ with $R_P = T_P \implies D \not\subset P$.
   (d) If $R'$ is a f.g. $R$-module and $P \in \text{Spec}(R)$, then: $R_P$ is integrally closed $\iff (R :_RR') \not\subset P$. 
(6) Let $R$ be an integral domain with quotient field $K$. Suppose that every ring between $R$ and $K$ is Noetherian. Prove that the Krull dimension of $R$ is at most 1. (This is the converse of Theorem 93.)

(7) Let an integral domain $R = \bigcap_i R_i$ be a locally finite intersection of one-dimensional local domains lying between $R$ and its quotient field. For each $i$, let $M_i$ denote the maximal ideal of $R_i$, and let $P_i = M_i \cap R$.

(a) Prove that any non-zero element of $R$ lies in only a finite number of minimal prime ideals of $R$.

(b) Assume $ht(P_i) = 1$ for each $i$. Prove that for any $0 \neq a \in R$, we have

$$Z(R/(a)) = P_1 \cup \ldots \cup P_n$$

where $P_1, \ldots, P_n$ are the minimal prime ideals containing $a$.

(c) Prove that every prime ideal of grade 1 has height 1.

(Part 2 – 30/100)

Solve the following problem.

Let $D$ be an integral domain with quotient field $K$ and $E$ a subset of $K$.

- Let $\text{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$ called the ring of $D$-integer-valued polynomials over $E$. If $E = D$, we write $\text{Int}(D)$ instead of $\text{Int}(D, D)$.

- The polynomial closure of $E$ in $D$ is the largest subset $F$ of $K$ with $\text{Int}(E, D) = \text{Int}(F, D)$ and is given by $\text{cl}_D(E) = \{x \in K \mid f(x) \in D \text{ for each } f \in \text{Int}(E, D)\}$.

- $E$ is said to be $D$-fractional if $\exists 0 \neq d \in D$ such that $dE \subseteq D$. So $dX \in \text{Int}(E, D)$.

- $E$ is polynomially dense in $D$ if $\text{Int}(E, D) = \text{Int}(D)$. (e.g., $N$ is polynomially dense in $\mathbb{Z}$.)

Assume that for some $\Delta \subseteq \text{Spec}(D)$, $D = \bigcap_{p \in \Delta} D_p$ is a locally finite representation of $D$. Then prove the following:

(a) $\text{Int}(E, D)_p = \text{Int}(E, D_p)$, for each $p \in \Delta$.

(b) $\text{cl}_D(E) = \bigcap_{p \in \Delta} \text{cl}_D(E_p)$.

(c) If $E \subseteq D$, then: $E$ is polynomially dense in $D \iff E$ is polynomially dense in $D_p \forall p \in \Delta$. 