Math 102 Quiz 7 13/04/2013

Decide whether the following improper integrals converge or diverge:

a) $I = \int_{\pi}^{+\infty} \frac{2+\cos x}{x}\,dx$, b) $J = \int_{1}^{+\infty} \frac{dx}{x^3+1}$

c) $K = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{x^4+1}}$

Solution:

a) For all $x \geqslant \pi$, $\frac{2+\cos x}{x} \geqslant \frac{1}{x}$

$$\int_{\pi}^{+\infty} \frac{dx}{x} = \lim_{t \to +\infty} \int_{\pi}^{t} \frac{dx}{x} = \lim_{t \to +\infty} \left[\ln t - \ln \pi\right] = +\infty$$

Since $I \geqslant \int_{\pi}^{+\infty} \frac{dx}{x}$ and since $\int_{\pi}^{+\infty} \frac{dx}{x}$ diverges, we conclude that I diverges

b) For all $x \geqslant 1$, $\frac{1}{x^3+1} \leqslant \frac{1}{x^3}$.

$$\int_{1}^{+\infty} \frac{dx}{x^3} = \lim_{t \to +\infty} \int_{1}^{t} \frac{dx}{x^3} = \lim_{t \to +\infty} \left[\frac{-1}{2x^2}\right]_1^t$$

$$= \lim_{t \to +\infty} -\frac{1}{2}\left(\frac{1}{t^2} - 1\right) = \frac{1}{2}.$$

Since $J \leqslant \int_{1}^{+\infty} \frac{dx}{x^3}$ and $\int_{1}^{+\infty} \frac{dx}{x^3}$ converges, we conclude that J converges.

c) $K = 2\int_0^{+\infty} \frac{dx}{\sqrt{x^4+1}} = 2\left[\int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{+\infty} \frac{dx}{\sqrt{x^4+1}}\right]$.

Now: For $x \geqslant 1$, $\frac{1}{\sqrt{x^4+1}} \leqslant \frac{1}{x^2}$. Since $\int_1^{+\infty} \frac{dx}{x^2}$ converges, we see that $\int_1^{+\infty} \frac{dx}{\sqrt{x^4+1}}$ converges

On the other hand the function $x \longmapsto \frac{1}{\sqrt{x^4+1}}$ is continuous on the closed interval $[0,1]$. Hence $\int_0^1 \frac{dx}{\sqrt{x^4+1}}$ is a finite real number.

Therefore K converges.