King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 201  Major Exam I
The Second Semester of 2012-2013 (122)
Time Allowed: 120 Minutes

Name: ___________________________ ID#: ___________________________
Section/Instructor: ________________ Serial #: ___________________________

- Mobiles and calculators are not allowed in this exam.
- Provide all necessary steps required in the solution.

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Q:1 Consider the parametric equations \( x = 4 \sin t, \ y = -5 \cos t \).

(a) (4 points) Eliminate the parameter to find a cartesian equation.

(b) (8 points) Sketch the curve for \(- \pi \leq t \leq \frac{\pi}{2}\) and mark the direction in which the curve is traced as \(t\) increases.

\[
\begin{align*}
(a) \quad \frac{x}{4} &= \sin t \quad \text{and} \quad \frac{y}{-5} = \cos t \ . \\
\text{Then} \quad \frac{x^2}{16} + \frac{y^2}{25} &= \sin^2 t + \cos^2 t = 1
\end{align*}
\]

(b) \[
\begin{array}{|c|c|c|}
\hline
x & y \\
\hline
\pi & 0 & 5 \\
-\pi & 0 & 5 \\
\frac{\pi}{2} & 4 & 0 \\
0 & 0 & -5 \\
\frac{\pi}{2} & 4 & 0 \\
\hline
\end{array}
\]
Q: 2  (a) (8 points) At what point(s) on the curve \( x = t^2 + 4t, \ y = 6t^2 \) is the tangent parallel to the line with parametric equations \( x = -3t, \ y = 12t - 5 \)?

\[
\frac{dx}{dt} = 2t + 4 \quad \frac{dy}{dt} = 12t
\]

The curve has slope \( \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t}{2t+4} \) (i)

The line with parametric equations
\[
\begin{align*}
x &= -3t, \\
y &= 12t - 5
\end{align*}
\]

Slope of line is \(-4\). (ii)

From (i) and (ii), we have
\[
\frac{12t}{2t+4} = -4 \quad \Rightarrow \quad \frac{3t}{2t+4} = -1
\]

\[
\Rightarrow \quad 5t + 4 = 0 \quad \Rightarrow \quad t = -\frac{4}{5}
\]

Point on the curve is \((x, y)_t = \left(-\frac{64}{25}, \frac{96}{25}\right)\).

(b) (4 points) At what point(s) on the curve of part (a) is the tangent vertical?

\[
\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} \neq 0
\]

\[
2t + 4 = 0
\]

\[
\Rightarrow \quad t = -2
\]

Point is \((0, 24)\).
Q:3 (12 points) Find the length of the curve

\[ x = 8 \cos t + 8t \sin t, \quad y = 8 \sin t - 8t \cos t; \quad 0 \leq t \leq \frac{\pi}{2}. \]

**Sol.**

\[ \frac{dx}{dt} = -8 \sin t + 8 \sin t + 8 \cos t = 8 \cos t \]

\[ \frac{dy}{dt} = 8 \cos t - 8 \cos t + 8 \sin t = 8 \sin t \]

**Length**

\[ \text{Length} = \int_{0}^{\pi/2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \ dt \]

\[ = \int_{0}^{\pi/2} \sqrt{64 \cos^2 t} \ dt \]

\[ = \int_{0}^{\pi/2} 8 \cos t \ dt \]

\[ = 4 \left( \sin t \right)_{0}^{\pi/2} \]

\[ = \pi^2 \]
Q:4 (a) (8 points) Write the polar equation $r = -8 \cos \theta$ in cartesian coordinates.

\[ r = -8 \cos \theta \]
\[ r^2 = -8r \cos \theta \]
\[ x^2 + y^2 = -8x \]
\[ \Rightarrow x^2 + y^2 + 8x = 0 \]
\[ \Rightarrow x^2 + 8x + 16 + y^2 = 16 \]
\[ \Rightarrow (x + 4)^2 + (y - 0)^2 = 4^2. \]

(b) (6 points) Sketch the graph of the resulting equation in Part(a).

\[ y \]
\[ (-4, 0) \]

Circle
Q: 5 (14 points) Find the area of the region that lies inside both curves \( r = \cos 2\theta \) and \( r = \sqrt{3} \sin 2\theta \) for \( 0 \leq \theta \leq \frac{\pi}{2} \).

**Sol.** The curves intersect at 
\[
\sqrt{3} \sin 2\theta = \cos 2\theta \\
\tan 2\theta = \frac{1}{\sqrt{3}} \\
2\theta = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12}
\]

Area of the region = \( \frac{1}{2} (A_1 + A_2) \)

Where 
\[
A_1 = \int_{0}^{\frac{\pi}{12}} \frac{1}{2} r^2 \, d\theta = \frac{3}{2} \int_{0}^{\frac{\pi}{12}} \sin^2 2\theta \, d\theta
\]

\[
= \frac{3}{4} \int_{0}^{\frac{\pi}{12}} (1 - \cos 4\theta) \, d\theta
\]

\[
= \frac{3}{4} \left[ \theta - \frac{\sin 4\theta}{4} \right]_{0}^{\frac{\pi}{12}} = \frac{3}{4} \left( \frac{\pi}{12} - \frac{\sqrt{3}}{8} \right)
\]

\[
A_2 = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{1}{2} (\cos 2\theta)^2 \, d\theta = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \left( \frac{1 + \cos 4\theta}{4} \right) \, d\theta
\]

\[
= \frac{1}{4} \left[ \theta + \frac{\sin 4\theta}{4} \right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = \frac{1}{4} \left[ \frac{\pi}{4} - \frac{\pi}{12} - \frac{\sqrt{3}}{8} \right]
\]

\[
= \frac{\pi}{24} - \frac{\sqrt{3}}{32}
\]

Area = \( \frac{3\pi}{48} + \frac{\pi}{24} - \frac{4\sqrt{3}}{32} \)

\[
= \frac{5\pi}{48} - \frac{4\sqrt{3}}{32}
\]
Q: 6 (10 points) Find an equation of the sphere that passes through the point \((2, -4, 3)\) and has center \((1, 2, 5)\). Describe the intersection of this sphere with the \(xz\)-plane.

Solution:

Radious = \(\sqrt{(1-2)^2 + (2+4)^2 + (5-3)^2}\)

\[= \sqrt{1+36+4} = \sqrt{41}\]

Equation of sphere is \((x-1)^2 + (y-2)^2 + (z-5)^2 = 41\)

The intersection of this sphere with \(xz\)-plane is the set of all points on the sphere whose \(y\)-coordinate is zero.

Putting \(y = 0\) in the eqn of sphere, we get \((x-1)^2 + (z-5)^2 = 37\), which represents a circle in the \(xz\)-plane with centre \((1, 0, 5)\) and radius \(\sqrt{37}\).
Q:7 (8 points) Let \( \vec{a} = <1, 1, 1> \) and \( \vec{b} = <2, 3, 4> \). Find \( \text{comp}_\vec{a} \vec{b} \) and \( \text{proj}_\vec{a} \vec{b} \).

\[
\text{comp}_\vec{a} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4}{\sqrt{1+1+1}} = \frac{9}{\sqrt{3}}
\]

\[
\text{proj}_\vec{a} \vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \left( \frac{\vec{a}}{|\vec{a}|} \right) = \frac{9}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot <1, 1, 1> = <3, 3, 3>
\]

(b) (6 points) Show that, in general, if \( \vec{u}, \vec{v} \) are non-zero vectors, then \( \vec{u} - \text{proj}_\vec{v} \vec{u} \) is orthogonal to \( \vec{v} \).

\[
\text{Sal.} \quad (\vec{u} - \text{proj}_\vec{v} \vec{u}) \cdot \vec{v} = \vec{u} \cdot \vec{v} - \text{proj}_\vec{v} \vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} (\vec{v} \cdot \vec{v}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0
\]

\( \Rightarrow \vec{u} - \text{proj}_\vec{v} \vec{u} \) is orthogonal to \( \vec{v} \).
Q:8 (12 points) If the points $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$, $D(0, 1, k)$ are in the same plane, then find the value of $k$.

\[
\begin{align*}
\overrightarrow{AB} &= \langle -1, 2, 0 \rangle \\
\overrightarrow{AC} &= \langle -1, 0, 3 \rangle \\
\overrightarrow{AD} &= \langle -1, 1, k \rangle
\end{align*}
\]

The volume of the parallelepiped formed from these vectors must be zero. Then

\[
\begin{vmatrix}
-1 & 2 & 0 \\
-1 & 0 & 3 \\
-1 & 1 & k
\end{vmatrix} = 0
\]

\[-1(-3) - 2(-k+3) = 0 \]

\[3 + 2k - 6 = 0 \]

\[2k = 3 \]

\[k = \frac{3}{2} \]