King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 201
Final Exam – 2012–2013 (122)
Allowed Time: 180 minutes

Instructions:
1. Write clearly and legibly. You may lose points for messy work.
2. Show all your work. No points for answers without justification.
3. Calculators and Mobiles are not allowed.

Part I: Written Problems

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Part II: MCQ Problems

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Q:1 (14 points) Find the critical points of the function

\[ f(x, y) = 7x + 4x^2 + y^2 + 2xy^2 + y^4. \]

Classify each point as local maximum, local minimum or saddle point.

\[
\begin{align*}
f_x &= 7 + 8x + 2y^2 \\
\therefore f_{xx} &= 8 \\
f_y &= 2y + 4xy + 4y^3 \\
\therefore f_{yy} &= 2 + 4x + 12y^2 \\
f_{xy} &= 4y
\end{align*}
\]

\( f_y = 0 \Rightarrow y = 0 \) or \( 2y^2 = -2x - 1 \)

At \( y = 0 \), \( f_x = 0 \Rightarrow x = -\frac{7}{8} \)

At \( 2y^2 = -2x - 1 \), \( f_x = 0 \Rightarrow 7 + 8x - 2x - 1 = 0 \Rightarrow x = -1 \)

\( \frac{y}{2} = \pm \frac{1}{\sqrt{2}} \)

Critical points are \( (-\frac{7}{8}, 0), (-1, -\frac{1}{\sqrt{2}}), (-1, \frac{1}{\sqrt{2}}) \)

At \( (-1, \frac{1}{\sqrt{2}}) \):

\[
f_{xx} f_{yy} - (f_{xy})^2 = (8)(4) - 8 > 0
\]

\( (-1, \frac{1}{\sqrt{2}}) \) Local minimum

At \( (-1, -\frac{1}{\sqrt{2}}) \):

\[
f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0 \quad \text{Local min.}
\]

At \( (-\frac{7}{8}, 0) \):

\[
f_{xx} f_{yy} - (f_{xy})^2 = (8)(2 - \frac{7}{2}) - 0 = -12 < 0
\]

\( \Rightarrow (-\frac{7}{8}, 0) \) is a saddle point.
Q2 (14 points) Use Lagrange multipliers to find the maximum and minimum values of

\[ f(x, y, z) = x^2y^2z^2 \]

subject to the constraint \( x^2 + y^2 + z^2 = 1 \).

\[ \text{Sal.} \quad \nabla f = \lambda \nabla g \quad \text{with} \quad x^2 + y^2 + z^2 = 1. \]

\[ \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2yz^2x^2 \rangle \]

\[ \nabla (x^2 + y^2 + z^2) = \langle 2x, 2y, 2z \rangle \]

\[ \nabla f = \lambda \nabla g \Rightarrow \]

\[ 2xy^2z^2 = \lambda 2x \]

\[ 2yx^2z^2 = \lambda 2y \]

\[ 2yz^2x^2 = \lambda 2z \]

If \( x, y, z \) are all \( \neq 0 \), then

\[ y^2z^2 = x^2z^2 = x^2y^2 = \lambda \]

\[ \Rightarrow x^2 = y^2 = z^2 \]

Now \( x^2 + y^2 + z^2 = 1 \) \( \Rightarrow 3x^2 = 1 \) \( \Rightarrow x^2 = \frac{1}{3} \)

So \( y^2 = \frac{1}{3}, \ z^2 = \frac{1}{3} \) & \( f(x, y, z) = \frac{1}{27} \)

If one of \( x, y, z \) is zero, then \( f(x, y, z) = 0 \)

Maximum value = \( \frac{1}{27} \)

Minimum value = 0
Q:3 (12 points) Evaluate the iterated integral

\[
\int_0^4 \int_0^{\sqrt{x}} \frac{1}{y^3 + 1} \, dy \, dx.
\]

\[
\begin{align*}
\text{Set: } & \hspace{1cm} \int_0^4 \int_0^{\sqrt{x}} \frac{1}{y^3 + 1} \, dy \, dx \\
& = \int_0^2 \left[ \frac{1}{y^3 + 1} \right]_{y=0}^{y=2} \, dx \\
& = \int_0^2 \frac{y^2}{y^3 + 1} \, dy \\
& = \frac{1}{3} \ln(y^3 + 1) \bigg|_0^2 \\
& = \frac{1}{3} \left[ \ln 9 - \ln 1 \right] = \frac{1}{3} \ln 9
\end{align*}
\]
Q: 4 (14 points) Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 36 - 3x^2 - 3y^2 \).

**Solution:**

The projection of the solid in the \( xy \)-plane is a circle with radius given by solving the equation

\[
x^2 + y^2 = 36 - 3x^2 - 3y^2
\]

or

\[
x^2 + y^2 = 9
\]

The volume can be found as follows:

\[
\text{Volume} = \iiint_E dV = \int_0^{2\pi} \int_0^3 \int_0^{36-3r^2} rdzdrd\theta
\]

\[
= \int_0^{2\pi} \int_0^3 \left[ z \right]_0^{36-3r^2} rd\theta dr
\]

\[
= \int_0^{2\pi} \int_0^3 r(36 - 4r^2) dr d\theta
\]

\[
= \left[ \frac{18r^2 - r^4}{2} \right]_0^3 d\theta
\]

\[
= \int_0^{2\pi} 81 d\theta
\]

\[
= 162\pi
\]
Q: 5 (16 points) Use spherical coordinates to evaluate

\[ \iiint_E (x^2 + y^2) \, dV, \]

where \( E \) is the region bounded above by the sphere \( x^2 + y^2 + z^2 = 1 \) and below by the cone \( z = \frac{1}{\sqrt{3}} \sqrt{x^2 + y^2} \)

Sal.: \[ x^2 + y^2 + z^2 = 1 \quad \Leftrightarrow \quad \rho = 1 \]

\[ \rho \cos \phi = \frac{1}{\sqrt{3}} \rho \sin \phi \quad \Rightarrow \quad \tan \phi = \sqrt{3} \Rightarrow \phi = \frac{\pi}{3} \]

\[ \iiint_E (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \left( \rho^2 \sin^2 \phi \right) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

\[ = 2\pi \int_0^{\pi/3} \sin^3 \phi \, d\phi \int_0^1 \rho^4 \, d\rho \]

\[ = \frac{2\pi}{5} \int_0^{\pi/3} \sin^3 \phi \, d\phi \]

\[ = \frac{2\pi}{5} \int_0^{\pi/3} (1 - \cos^2 \phi) \sin \phi \, d\phi \]

\[ = \frac{2\pi}{5} \left[ -\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^{\pi/3} \]

\[ = \frac{2\pi}{5} \left[ -\frac{1}{2} + \frac{1}{24} + 1 - \frac{1}{3} \right] \]

\[ = \frac{2\pi}{5} \cdot \frac{5}{24} = \frac{\pi}{12} \]
Q:6 (7 points) The slope of the tangent line to the polar curve $r = 1 + 2 \cos \theta$ at the point $\theta = \pi/3$ is

(A) $\frac{\sqrt{3}}{3}$

(B) $\frac{1}{\sqrt{3}}$

(C) $\frac{1}{3}$

(D) $3\sqrt{3}$

(E) $-\frac{\sqrt{3}}{9}$

Q:7 (7 points) The area of the region that lies inside both curves $r = \cos \theta$ and $r = \sin \theta$ is

(A) $\frac{1}{8}(\pi - 2)$

(B) $\frac{1}{8}(\pi + 2)$

(C) $-\frac{1}{8}(\pi - 2)$

(D) $-\frac{1}{8}(\pi + 2)$

(E) $\frac{1}{8}(2\pi - 2)$
Q:8 (7 points) The area of the surface obtained by rotating the curve parametrized by 
\[ x = 3t - t^3, \quad y = 3t^2, \quad 0 \leq t \leq 1 \] about the \( x \)-axis is

(A) \( \frac{48\pi}{5} \)

(B) \( \frac{18\pi}{15} \)

(C) \( -\frac{24\pi}{15} \)

(D) \( -\frac{16\pi}{15} \)

(E) \( \frac{\pi}{15} \)

Q:9 (7 points) The value of \( k \) for which the vectors \( \vec{a} = \langle 1, 4, -7 \rangle, \vec{b} = \langle 4, 0, 2 \rangle \) and 
\( \vec{c} = \langle k, 0, 1 \rangle \) are coplanar is

(A) 2

(B) \( -2 \)

(C) 1

(D) \( -1 \)

(E) 0
Q: 10 (7 points) Where does the line that passes through (1, 0, 1) and (4, -2, 2) intersect the plane \( x + y + z = 6 \) ?

(A) (7, -4, 3)  
(B) (1, 2, 3)  
(C) (3, 4, -1)  
(D) (-1, 4, 3)  
(E) (6, 4, -4)

Q: 11 (7 points) The equation \( x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 = 0 \) represents

(A) a cone  
(B) a hyperboloid of two sheets  
(C) a hyperboloid of one sheet  
(D) an elliptic paraboloid  
(E) a sphere
Q:12 (7 points) Consider the surface

\[ x^2 z + 3yz^2 + 3xyz = 7. \]

Let \( 5x + By + Cz = D \) be an equation of the tangent plane to the given surface at \((1, 1, 1)\). The value of \( B + C + D \) is

(A) 37  
(B) 32  
(C) 34  
(D) 42  
(E) 39

Q:13 (7 points) The maximum rate of change of \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) at the point \((1, 1, 1)\) is

(A) 1  
(B) 3  
(C) 4  
(D) 5  
(E) 6
Q:14 (7 points) Consider

\[ f(x, y) = \begin{cases} \frac{x^2y^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases} \]

(A) \( f(x, y) \) has a removable discontinuity at \((0, 0)\)

(B) \( \lim_{(x,y)\to(0,0)} f(x, y) \) does not exist.

(C) \( f(x, y) \) is continuous at \((0, 0)\)

(D) \( f(x, y) \) is continuous everywhere

(E) \( f(x, y) \) is continuous

Q:15 (7 points) If \( u = x^4y + y^2z^3 \), where \( x = r se^t, y = rs^2 e^{-t} \) and \( z = r^2 s \) sint, then the value of \( \frac{\partial u}{\partial s} \) when \( r = 1, s = 1, t = 0 \) is

(A) 6

(B) 30

(C) 4

(D) 25

(E) 10