King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

EXAM I – MATH 202 (Term 122)
February 26, 2013

Duration: 120 Minutes

Name: Solution key ID#: 
Section/Instructor: Serial #: 

- Provide all necessary steps with clear writing.
- Mobiles and calculators are NOT allowed in this exam.

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Q1. Consider the differential equation \( (y^2 + y) - \frac{dy}{dx} = 0 \).

(a) \textbf{(5 points)} Verify that \( y = \frac{Ce^x}{1 - Ce^x} \) is a one-parameter family of solutions of the given differential equation.

\[
\text{Solution: } \quad y = \frac{Ce^x}{1 - Ce^x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{Ce^x}{(1 - Ce^x)^2}
\]

\[
\Rightarrow \quad y^2 + y = \frac{C^2 e^{2x}}{(1 - Ce^x)^2} + \frac{Ce^x}{1 - Ce^x}
\]

\[
= \frac{C^2 e^{2x} + Ce^x (1 - Ce^x)}{(1 - Ce^x)^2} = \frac{-Ce^x}{(1 - Ce^x)^2}
\]

Hence,

\[
y^2 + y - \frac{dy}{dx} = \frac{Ce^x}{(1 - Ce^x)^2} - \frac{Ce^x}{(1 - Ce^x)^2} = 0.
\]

Therefore, \( y = \frac{Ce^x}{1 - Ce^x} \) is a one-parameter family of solutions of the given DE.

(b) \textbf{(4 points)} Find two constant solutions of the given differential equation.

\[
\text{Solution: Constant solutions can be obtained by solving the equation}
\]

\[
y^2 + y = 0.
\]

We have \( y = 0 \) or \( y = -1 \).

(c) \textbf{(4 points)} Find a singular solution of the given differential equation.

\[
\text{Solution: \quad } \begin{cases} 
\text{\( y = 0 \) can be determined from the family in (a) by choosing } C = 0 \\
\text{\( y = -1 \) cannot be obtained from the family.}
\end{cases}
\]

Thus, \( y = -1 \) is a singular solution of the given DE.
Q2. (7 points) Find the value of \( b \) such that the initial value problem

\[
\frac{dy}{dx} = \sqrt{y - 3x}, \quad y(2) = b
\]

has a unique solution.

Solution:

Here \( f(x,y) = \sqrt{y - 3x} \) and \( f_y = \frac{1}{2\sqrt{y - 3x}} \).

Hence \( f \) and \( f_y \) are both continuous when \( y - 3x > 0 \) \( \Rightarrow \) \( y > 3x \).

For \( x_0 = 2 \), we must have \( y_0 > 3(2) = 6 \).

Therefore \( b > 6 \).
Q3. (10 points) Solve the separable differential equation

\[(3x + y + xy + 3)dx + (x^2 + 2x)dy = 0.\]

**Solution:**

The given equation can be written as

\[
\left[ 3(x+1) + y(x+1) \right] dx + (x^2 + 2x) dy = 0
\]

\[\Rightarrow (3+y)(x+1) dx + (x^2 + 2x) dy = 0\]

\[\Rightarrow \frac{x+1}{x^2 + 2x} dx + \frac{dy}{3+y} = 0\]

\[\Rightarrow \left( \frac{1}{x} + \frac{1}{x+2} \right) dx + \frac{dy}{3+y} = 0\]

\[\Rightarrow \frac{1}{2} \ln |x| + \frac{1}{2} \ln |x+2| + \ln |3+y| = C_1\]

\[\Rightarrow (3+y)\sqrt{x(x+2)} = C\]
Q4. Consider the linear differential equation

\[ xy' + (1 + x)y = e^{-x} \cos^2 x. \]

(a) (14 points) Find a solution of the equation that passes through the point \((-\pi, 0)\).

\[ \text{Solution: The standard form of the equation is} \]
\[ y' + \left(\frac{1+x}{x}\right)y = \cos^2 x. \]

Here \( P(x) = \frac{1+x}{x} \). The integrating factor is
\[ e^{\int P(x) \, dx} = e^{\int \frac{1+x}{x} \, dx} = e^{\ln x + \ln x} = xe^x. \]

We obtain
\[ \mathcal{L}(xe^x y) = \cos^2 x \]
\[ = \frac{1}{2} \left(1 + \cos 2x\right) \]
\[ \Rightarrow xe^x y = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x\right) + C \]
\[ \Rightarrow y = \frac{1}{2e^x} + \frac{\sin 2x}{4xe^x} + \frac{C}{xe^x} \]

\[ y(-\pi) = 0 \]
\[ \Rightarrow \left(\frac{1}{2} + \frac{C}{-\pi}\right)e^{-\pi} = 0 \]
\[ \Rightarrow \frac{C}{\pi} = \frac{1}{2} \Rightarrow C = \frac{\pi}{2} \]

So, the solution is
\[ y = \frac{1}{2e^x} + \frac{\sin 2x}{4xe^x} + \frac{\pi}{2xe^x} \]

(b) (2 points) Give the largest interval in which the solution in (a) is defined.

\[ \text{Solution:} \]

The interval is
\[ I = (-\infty, 0) \]
Q5. (15 points) Solve the initial value problem

\[ ye^{2xy} + x + xe^{2xy} \frac{dy}{dx} = 0, \quad y(1) = 0 \]

**Solution:**

The given equation can be written as

\[ (ye^{2xy} + x) dx + xe^{2xy} dy = 0 \]

Put \( M = ye^{2xy} + x \)
\[ N = xe^{2xy} \]

We have \( My = e^{2xy} + 2xye^{2xy} \)
\[ Nx = e^{2xy} + 2xye^{2xy} \]

Since \( My = Nx \), the equation is an exact equation.

\[ \Rightarrow F(x,y) = \int M dx + g(y) \]
\[ = \int (ye^{2xy} + x) dx + g(y) \]
\[ = \frac{1}{2} e^{2xy} + \frac{1}{2} x^2 + g(y) \]

\[ \Rightarrow F_y = N \quad \Rightarrow \quad xe^{2xy} + g'(y) = xe^{2xy} \]
\[ \Rightarrow g'(y) = 0 \]
\[ \text{Put } g(y) = 0. \]

The general solution is \( \frac{1}{2} e^{2xy} + \frac{1}{2} x^2 = c_1 \)

\[ \Rightarrow e^{2xy} + x^2 = c \]

Imposing the initial condition \( y(1) = 0 \) gives \( c = 2 \).

Thus, the solution of the IVP is

\[ e^{2xy} + x^2 = 2 \]
Q6. (10 points) Find an integrating factor that makes the differential equation 
\[ xy^3 + y + (2x^2y^2 + 2x + 2y^4)y' = 0 \]
exact.
(Note: Do not solve the new equation)

Solution:

The equation can be written as
\[ (xy^3 + y)dx + (2x^2y^2 + 2x + 2y^4)dy = 0 \]

Put \( M = xy^3 + y \)
\( N = 2x^2y^2 + 2x + 2y^4 \).

We have \( M_y = 3xy^2 + 1 \)
\( N_x = 4xy^2 + 2 \).

\[ \frac{N_x - M_y}{M} = \frac{(4xy^2 + 2) - (3xy^2 + 1)}{xy^3 + y} \]
\[ = \frac{xy^2 + 1}{xy^3 + y} = \frac{1}{y} \]

The integrating factor is

\[ M(y) = e^{\int \frac{1}{y} dy} \]
\[ = e^{\ln y} = y \]
Q7. (a) (10 points) Use an appropriate substitution to reduce the differential equation
\[
\frac{dy}{dx} = y(xy^3 - 1)
\]
to a linear equation. (Note: Do not solve the new equation)

Solution: The equation can be written as
\[
\frac{dy}{dx} + y = xy^4.
\]
It is a Bernoulli equation with \( n = 4 \).

Put \( u = y^{1-n} = y^{-3} \) \( \Leftrightarrow y = u^{-\frac{1}{3}} \).

\[\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -\frac{1}{3} u^{-4/3} \frac{du}{dx}\]

Substituting \( u = y^{-3} \) gives
\[-\frac{1}{3} u^{-4/3} \frac{du}{dx} + u^{-1/3} = x u^{-1/3}\]

\[\Rightarrow \frac{du}{dx} - 3u = -3x,\]

(b) (7 points) Use a suitable substitution to reduce the differential equation
\[
\frac{dy}{dx} = \frac{x+y}{x+y+1}
\]
to a separable equation. (Note: Do not solve the new equation)

Solution: Put \( u = x+y \) \( \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx} \)

\[\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1\]

We obtain
\[\frac{du}{dx} - 1 = \frac{u}{u+1}\]

\[\Rightarrow \frac{du}{dx} = \frac{u}{u+1} + 1 = \frac{2u+1}{u+1}\]

\[\Rightarrow \frac{u+1}{2u+1} du = dx\]
Q8. (12 points) A thermometer reading 70°F is taken from inside a room to the outside, where the air temperature is 20°F. Four minutes later, the thermometer reads 30°F. Find the thermometer’s reading at \( t = 8 \) minutes.

\[ \text{Solution: Newton’s law of cooling/warming gives} \]
\[ \frac{dT}{dt} = k(T - T_m) . \]

In this case, \( T_m = 20 \).

We have \( \frac{dT}{dt} = k(T - 20) \) or \( \frac{dT}{T - 20} = k \, dt \)

\[ \Rightarrow \quad T(t) = 20 + Ce^{kt} . \]

We know that \( T(0) = 70 \),

\[ \Rightarrow \quad 70 = 20 + C \quad \Rightarrow \quad C = 50 , \]

and \( T(4) = 30 \)

\[ \Rightarrow \quad 30 = 20 + 50e^{4k} \]

\[ \Rightarrow \quad e^{4k} = \frac{1}{5} \]

\[ \Rightarrow \quad k = - \frac{\ln 5}{4} . \]

Thus,

\[ T(t) = 20 + 50e^{\frac{-\ln 5}{4} t} . \]

and \( T(8) = 20 + 50e^{\frac{-2 \ln 5}{4} \times 8} \)

\[ = 20 + 50 \left( \frac{1}{25} \right) \]

\[ = 22 . \]

The thermometer’s reading after 8 minutes is 22°F.