King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

FINAL EXAM – MATH 202 (Term 122)
May 27, 2013

Duration: 180 Minutes

Name : SOLUTION_KEY
ID# : 
Section #: 
Serial #: 

Please read the following:

1. Exam has 2 parts: Part I: 10 MCQs, Part II: 5 Written Questions.
2. Provide all necessary steps with clear writing for Part II.
3. For Part I (MCQ), credit will be given only for the correct answer posted BELOW.
4. Mobiles and calculators are NOT allowed in this exam.

Part I: Multiple Choice [7 Points for each correct answer]

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<th>Student Answer</th>
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Part II: Written

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Grand Total /140
Part I: Multiple Choice

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Part II: Written

Q11. (14 points) Find two power series solutions of the differential equation
\[ y'' + x^2 y' + xy = 0 \]
about the ordinary point \( x = 0 \). Give the first three nonzero terms for each series solution.

**Hint:** The assumption \( y = \sum_{n=0}^{\infty} c_n x^n \) and its first two derivatives lead to

\[ \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=2}^{\infty} (k-1) c_{k-1} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k = 0. \]  

(1)

**Solution:** Equation (1) can be written as

\[ 2c_2 + 6c_3 x + c_0 x + \sum_{k=2}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=2}^{\infty} (k-1) c_{k-1} x^k + \sum_{k=2}^{\infty} c_{k-1} x^k = 0 \]

or \( 2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} [(k+2)(k+1)c_{k+2} + k c_{k-1}] x^k = 0. \)

We have \( c_2 = 0 \), \( c_3 = -c_0 / 6 \) and

\[ c_{k+2} = -\frac{kc_{k-1}}{(k+2)(k+1)}, \quad k = 2, 3, 4, \ldots \]  

(2)

This relation generates consecutive coefficients of the assumed solution as we let \( k \) take on the successive integers indicated in (2):

\[ k = 2, \quad c_4 = -\frac{2c_2}{4 \cdot 3} = -\frac{c_2}{6} \]

\[ k = 3, \quad c_5 = -\frac{3c_3}{5 \cdot 4} = 0 \]

\[ k = 4, \quad c_6 = -\frac{4c_4}{6 \cdot 5} = \frac{4c_0}{6^2 \cdot 5} \]

\[ k = 5, \quad c_7 = -\frac{5c_5}{7 \cdot 6} = \frac{5c_1}{6^2 \cdot 7} \]

and so on. Now we substitute the coefficients just obtained into the original assumption

\[ y = c_0 + c_1 x - \frac{c_0}{6} x^3 - \frac{c_1}{6} x^4 + \frac{4c_0}{6^2 \cdot 5} x^6 + \frac{5c_1}{6^2 \cdot 7} x^7 - \cdots \]

\[ = c_0 y_1(x) + c_1 y_2(x), \]

where

\[ y_1(x) = 1 - \frac{1}{6} x^3 + \frac{4}{6^2 \cdot 5} x^6 - \cdots \]

\[ y_2(x) = x - \frac{1}{6} x^4 + \frac{5}{6^2 \cdot 7} x^7 - \cdots \]
Q12. (14 points) Determine singular points of the differential equation

\[ x^2(x^2 - 1)^2y'' + 2x(x - 1)y' + y = 0. \]

Classify each singular point as regular or irregular.

**Solution:**

It should be clear that \( x = 0, \ x = 1, \ x = -1 \) are singular points of the equation.

The standard form of the equation is

\[ y'' + P(x)y' + Q(x)y = 0, \]

where

\[ P(x) = \frac{2}{x(x - 1)(x + 1)^2} \quad \text{and} \quad Q(x) = \frac{1}{x^2(x + 1)^2(x - 1)^2}. \]

Since both rational functions

\[ p(x) = xP(x) = \frac{2}{(x - 1)(x + 1)^2} \quad \text{and} \quad q(x) = x^2Q(x) = \frac{1}{(x + 1)^2(x - 1)^2} \]

are analytic at \( x = 0 \). We conclude that \( x = 0 \) is a *regular singular* point.

Similarly, we are led to the conclusion that \( x = 1 \) is a *regular singular* point. This follows from the fact that

\[ p(x) = (x - 1)P(x) = \frac{2}{x(x + 1)^2} \quad \text{and} \quad q(x) = (x - 1)^2Q(x) = \frac{1}{x^2(x + 1)^2} \]

are both analytic at \( x = 1 \).

For \( x = -1 \),

\[ p(x) = (x + 1)P(x) = \frac{2}{x(x - 1)(x + 1)} \]

is not analytic at \( x = -1 \). Thus, \( x = -1 \) is an *irregular singular* point.
Q13. (14 points) Solve the initial value problem
\[
\frac{dx}{dt} = x + y \\
\frac{dy}{dt} = -2x - y, \quad x(0) = 1, \ y(0) = 1.
\]

**Solution:**

The characteristic equation of the system is
\[
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ -2 & -1 - \lambda \end{vmatrix} = \lambda^2 + 1 = 0.
\]

We find the eigenvalues \( \lambda_1 = i \) and \( \lambda_2 = -i \).

For \( \lambda_1 = i \) the system
\[
\begin{pmatrix} 1 - i & 1 \\ -2 & -1 - i \end{pmatrix} \begin{pmatrix} 1 - \lambda \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
gives \( k_2 = -(1 - i)k_1 \). By choosing \( k_1 = 1 \), we get
\[
K_1 = \begin{pmatrix} 1 \\ -1 + i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

We form the vectors
\[
B_1 = \text{Re}(K_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad B_2 = \text{Im}(K_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The general solution is
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t.
\]

Imposing the initial conditions \( x(0) = 1, \ y(0) = 1 \) gives \( c_1 = 1 \) and \( c_2 = 2 \).

Thus, the required solution is
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t + 2 \sin t \\ \cos t - 3 \sin t \end{pmatrix}.
\]
Q14. (16 points) Find the general solution of the system

\[ X' = \begin{pmatrix} 3 & 4 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix} X. \]

**Solution:** From the characteristic equation

\[
\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 4 & 0 \\ -1 & -\lambda & 2 \\ 0 & 2 & 3 - \lambda \end{vmatrix} = \lambda(3 - \lambda)^2 = 0,
\]

we see that the eigenvalues are \( \lambda_1 = 0 \) and \( \lambda_2 = 3 \) (of multiplicity two).

For \( \lambda_1 = 0 \) Gauss-Jordan elimination gives

\[
(A + 0I|0) = \begin{pmatrix} 3 & 4 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \end{pmatrix} \quad \text{row operations} \quad \begin{pmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

Therefore \( k_1 = \frac{4}{3}k_2, \quad k_3 = -\frac{2}{3}k_2 \). The choice \( k_2 = -3 \) gives an eigenvector

\[ K_1 = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}. \]

Similarly, for \( \lambda_2 = 3 \)

\[
(A + 3I|0) = \begin{pmatrix} 0 & 4 & 0 & 0 \\ -1 & -3 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \quad \text{row operations} \quad \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

implies that \( k_1 = 2k_3, \quad k_2 = 0 \). Choosing \( k_3 = 1 \) gives an eigenvector

\[ K_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \]

We next solve the system \((A + 3I)P = K_2:\)

\[
\begin{pmatrix} 0 & 4 & 0 & 2 \\ -1 & -3 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \quad \text{row operations} \quad \begin{pmatrix} 1 & 0 & -2 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

We find \( p_1 = 2p_3 - \frac{3}{2}, \quad p_2 = \frac{1}{2} \). Choosing \( p_3 = 0 \) gives

\[ P = \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}. \]

The general solution of the given equation is

\[ X(t) = c_1 \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} te^{3t} + \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} e^{3t}. \]
Q15. (12 points) Consider the nonhomogeneous system

\[ X' = AX + \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \]

If the general solution of the associated homogeneous system \( X' = AX \) is

\[ X_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t, \]

find a particular solution \( X_p \) of the nonhomogeneous system.

Solution:

The fundamental matrix of the system is given by

\[ \Phi(t) = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \]

and

\[ \Phi^{-1}(t) = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}. \]

Here \( F(t) = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \).

The required particular solution is

\[ X_p = \Phi(t) \int \Phi^{-1}(t)F(t)dt \]

\[ = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \int \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} dt \]

\[ = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \int \begin{pmatrix} -12 \\ 5e^{-t} \end{pmatrix} dt \]

\[ = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \begin{pmatrix} -12t \\ -15 \end{pmatrix} \]

\[ = \begin{pmatrix} -12t - 15 \\ -12t - 10 \end{pmatrix}. \]