King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

EXAM II – MATH 202 (Term 122)
April 06, 2013

Duration: 120 Minutes

Name: Solution Key ID#: 
Section #: Serial #: 

- Provide all necessary steps with clear writing.
- Mobiles and calculators are NOT allowed in this exam.

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Q1. (6 points) Find an interval centered about $x = 1$ for which the following initial value problem has a unique solution:

$$(x^2 - 4) y'' + 2xy' - 3y = 0, \; y(1) = 0, \; y'(1) = -1.$$ 

Solution:

1. We know that $a_2(x) = x^2 - 4, \; a_1(x) = 2x, \; a_0(x) = -3$, are all continuous on any interval containing $x = 1$.

2. $a_2(x) = 0$ iff $x = \pm 2$

Thus, the required interval is

$$I = (0, 2)$$
Q2. (a) (7 points) The functions \( y_1 = \sin(x^2) \) and \( y_2 = \cos(x^2) \) are both solutions of the differential equation

\[ xy'' - y' + 4x^3y = 0. \]

Verify that \( y_1 \) and \( y_2 \) form a fundamental set of solutions of the given equation on the interval \((0, \infty)\).

\[
\text{Solution: The Wronskian of } y_1, y_2 \text{ is } W(y_1, y_2) = \begin{vmatrix}
\sin(x^2) & \cos(x^2) \\
2x \cos(x^2) & -2x \sin(x^2)
\end{vmatrix}
= -2x \sin^2(x^2) - 2x \cos^2(x^2)
= -2x
\]

\[
W(y_1, y_2) \neq 0 \text{ iff } x \neq 0.
\]

Thus, the set \( y_1, y_2 \) is linearly independent on the interval \((0, \infty)\).

Since \( y_1, y_2 \) are solutions of the given DE and the set \( y_1, y_2 \) is linearly independent on \((0, \infty)\),

\( y_1, y_2 \) form a fundamental set of solutions of the DE on \((0, \infty)\).

(b) (6 points) Find a solution of the differential equation in part (a) that satisfies the boundary conditions \( y\left(\frac{\pi}{2}\right) = \sqrt{2} \) and \( y'\left(-\frac{\pi}{2}\right) = 0 \).

\[
\text{Solution: The general solution is } y = c_1 \sin(x^2) + c_2 \cos(x^2).
\]

\[
y' = 2c_1 x \cos(x^2) - 2c_2 x \sin(x^2).
\]

\[
y\left(\frac{\pi}{2}\right) = c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \cos\left(\frac{\pi}{4}\right) = \sqrt{2} \quad \Rightarrow \quad c_1 + c_2 = 2
\]

\[
y'\left(\frac{\pi}{2}\right) = 2c_1 \left(-\frac{\sqrt{2}}{2}\right) \sin\left(\frac{\pi}{4}\right) - 2c_2 \left(-\frac{\sqrt{2}}{2}\right) \cos\left(\frac{\pi}{4}\right) = 0
\]

\[
\Rightarrow \quad c_1 = c_2 = 1
\]

The solution is

\[ y = \sin(x^2) + \cos(x^2). \]
Q3. (a) (8 points) Verify that \( y_{p_1} = xe^x \) and \( y_{p_2} = -4x^2 \) are, respectively, particular solutions of
\[
y'' - 3y' + 4y = e^x(2x - 1) \quad \text{and} \quad y'' - 3y' + 4y = -16x^2 + 24x - 8.
\]

\underline{Solution}

\(2\) \( y_{p_1} = xe^x \), \( y_{p_1}' = e^x + xe^x \), \( y_{p_1}'' = 2e^x + xe^x \)
\[
y_{p_1}'' - 3y_{p_1}' + 4y_{p_1} = 2e^x + xe^x - 3e^x - 3xe^x + 4xe^x
\]
\[
= 2xe^x - e^x = e^x(2x - 1)
\]
So, \( y_{p_1} \) is a particular solution of \( y'' - 3y' + 4y = e^x(2x - 1) \)

\(2\) \( y_{p_2} = -4x^2 \), \( y_{p_2}' = -8x \), \( y_{p_2}'' = -8 \)
\[
y_{p_2}'' - 3y_{p_2}' + 4y_{p_2} = -8 + 24x - 16x^2
\]
So, \( y_{p_2} \) is a particular solution of \( y'' - 3y' + 4y = -16x^2 + 24x - 8 \)

(b) (5 points) Use part (a) to find a particular solution of
\[
y'' - 3y' + 4y = 2x^2 - 3x + 1 + 4xe^x - 2e^x.
\]

\underline{Solution}
The equation can be written as
\[
y'' - 3y' + 4y = 2e^x(2x - 1) - \frac{1}{8} (-16x^2 + 24x - 8).
\]
By the superposition principle (nonhomogeneous equations),
\[
y_p = 2y_{p_1} - \frac{1}{8} y_{p_2} = 2xe^x + \frac{x^2}{2}
\]
is a particular solution of the given equation.
Q4. (12 points) Given that $y_1(x) = x + 1$ is a solution of the differential equation

$$(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0,$$

find a second solution $y_2(x)$ of the equation.

\[\text{Solution:}\]

The standard form of the equation is

$$y'' + \frac{2(1+x)}{1-2x-x^2} y' - \frac{2}{1-2x-x^2} y = 0,$$

$$p(x) = \frac{2(1+x)}{1-2x-x^2}.$$

The second solution is

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{[y_1(x)]^2} \, dx$$

$$= (x+1) \int \frac{e^{-\int \frac{2(1+x)}{1-2x-x^2} \, dx}}{(x+1)^2} \, dx$$

$$= (x+1) \int \frac{\ln(x^2 + 2x - 1)}{(x+1)^2} \, dx$$

$$= (x+1) \int \frac{x^2 + 2x - 1}{(x+1)^2} \, dx$$

$$= (x+1) \left[ x + \frac{2}{x+1} \right]$$

$$= x^2 + x + 2.$$
Q5. (12 points) Solve the initial value problem

\[ y'''' - y''' + 9y'' - 9y = 0, \quad y(0) = 13, \quad y'(0) = 0, \quad y''(0) = 3. \]

**Solution:** The auxiliary equation is

\[ m^3 - m^2 + 9m - 9 = 0 \]

\[ (m - 1)(m^2 + 9) = 0 \]

\[ m_1 = 1, \quad m_2 = 3i, \quad m_3 = -3i \]

The general solution is

\[ y = c_1 e^x + c_2 \cos 3x + c_3 \sin 3x \]

\[ y' = c_1 e^x - 3c_2 \sin 3x + 3c_3 \cos 3x \]

\[ y'' = c_1 e^x - 9c_2 \cos 3x - 9c_3 \sin 3x \]

\[ y(0) = c_1 + c_2 = 13 \]

\[ y'(0) = c_1 + 3c_2 = 0 \]

\[ y''(0) = c_1 - 9c_2 = 0 \]

\[ \implies \begin{cases} c_1 = 12, \\ c_2 = 1, \\ c_3 = -4 \end{cases} \]

The solution is

\[ y = 12 e^x + \cos 3x - 4 \sin 3x \]
Q6. (15 points) Solve the differential equation
\[ y'' - y = x + \cos x \]
by undetermined coefficients (annihilator approach).

**Solution:**
From the auxiliary equation \( m^2 - 1 = 0 \), we have
\[ y_c = c_1 e^x + c_2 e^{-x} \]
Since \( D^2x = 0 \) and \( (D^2 + 1) \cos x = 0 \), we apply the differential operator \( D^4(D^2 + 1) \) to both sides of the equation:
\[ D^2(D^2 + 1)(D^2 - 1)y = 0 \]
The auxiliary equation of \( \circ \) is
\[ m^2(m^2 + 1)(m^2 - 1) = 0 \]
\[ \Rightarrow m_1 = 1, \ m_2 = -1, \ m_3 = i, \ m_4 = -i, \ m_5 = m_6 = 0 \]
Thus,
\[ y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + c_5 + c_6 x \]
After excluding the linear combination of terms corresponding to \( y_c \), we have
\[ y_p = A + B x + C \cos x + D \sin x \]
Substituting \( y_p \) in the given equation gives
\[ y_p'' - y_p = -A - B x - 2 C \cos x - 2 D \sin x = x + \cos x \]
Equating coefficients gives \( A = 0, \ B = -1, \ C = -\frac{1}{2} \) and \( D = 0 \).
We find \( y_p = -x - \frac{1}{2} \cos x \).
The general solution is
\[ y = c_1 e^x + c_2 e^{-x} - x - \frac{1}{2} \cos x \]
Q7. (16 points) Solve the differential equation

\[ y'' + 8y' + 16y = x^{-2}e^{-4x}, \quad x > 0 \]

by variation of parameters.

Solution. From the auxiliary equation

\[ m^2 + 8m + 16 = (m + 4)^2 = 0, \]

we have

\[ y_c = c_1e^{-4x} + c_2xe^{-4x}. \]

With the identifications \( y_1 = e^{-4x} \) and \( y_2 = xe^{-4x} \),

we next compute the Wronskian:

\[ W(y_1, y_2) = \begin{vmatrix} e^{-4x} & xe^{-4x} \\ -4xe^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = e^{-4x} - 4xe^{-4x} + 4xe^{-4x} = e^{-4x} \]

We identify \( f(x) = x^{-2}e^{-4x} \). We obtain

\[ W_1 = \begin{vmatrix} 0 & xe^{-4x} \\ -2xe^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = -xe^{-4x} \]

\[ W_2 = \begin{vmatrix} e^{-4x} & 0 \\ -4xe^{-4x} & xe^{-4x} \end{vmatrix} = xe^{-4x} \]

and

\[ u_1' = -x^{-1} \quad \Rightarrow \quad u_1 = -\ln x \]

\[ u_2' = x^{-2} \quad \Rightarrow \quad u_2 = -x^{-1}. \]

Thus,

\[ y_p = -xe^{-4x} \ln x - e^{-4x} \]

and the general solution is

\[ y = y_c + y_p = c_1e^{-4x} + c_2xe^{-4x} - e^{-4x} \ln x \]

\[ \square \]
Q8. (a) (6 points) Find a homogeneous linear differential equation with constant
coefficients for which \( y = c_1 e^{-2x} + c_2 e^{6x} \) is the general solution.

Solution: From the general solution we know
that the roots of the auxiliary equation
are \( m_1 = -2 \), \( m_2 = 6 \). This gives the
auxiliary equation

\[
(m + 2)(m - 6) = 0
\]

\[
\Rightarrow \quad m^2 - 4m - 12 = 0
\]

Thus, the required equation is

\[
y'' - 4y' - 12y = 0
\]

(b) (7 points) Use part (a) to find a nonhomogeneous linear differential
equation whose general solution is \( y = c_1 e^{-2x} + c_2 e^{6x} + x^2 + 2x \).

Solution: The nonhomogeneous equation
is of the form

\[
y'' - 4y' - 12y = g(x) \tag*{(*)}
\]

Substituting \( y_p = x^2 + 2x \) to (*) gives

\[
g(x) = y_p'' - 4y_p' - 12y_p
\]

\[
= 2 - 4(2x - 2) - 12(x^2 + 2x)
\]

\[
= -12x^2 - 32x - 6
\]

Thus, the required nonhomogeneous
equation is

\[
y'' - 4y' - 12y = -12x^2 - 32x - 6
\]