1. [20pts] (a) Prove that $x^2 + y^2 = 11(z^2 + w^2)$ has no nontrivial integer solution.
(b) Using the identity $(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$ determine all integer solutions of the equation $x^2 + y^2 = 2z^2$.
(c) Show that if $x^3 + 2y^3 + 4z^3 \equiv 6xyz \pmod{7}$ then $x \equiv y \equiv z \equiv 0 \pmod{7}$. Deduce that the equation $x^3 + 2y^3 + 4z^3 - 6xyz = 0$ has no nontrivial integer solutions.

2. [20pts] (a) Let the polynomial equation with integer coefficients $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$, where $c_n \neq 0$, have a nonzero rational solution $\frac{a}{b}$ with $a$ and $b$ coprime integers. Prove that $a|c_0$ and $b|c_n$.
(b) Let $a \in \mathbb{N}$. Prove that $\sqrt{a+1} - \sqrt{a}$ is irrational.
(c) Find, in terms of the prime $p$, the integer(s) $a$ for which $\sqrt{a+p} - \sqrt{a}$ is rational. What then are the possible values of the rational number $\sqrt{a+p} - \sqrt{a}$?

3. [20pts] (a) Let $a, b, x, y$ be integers such that $(a - b) \mid (ax + y)$. Prove that $(a - b) \mid (bx + y)$.
(b) Show that if $n$ is the product of odd distinct primes $p_1, \ldots, p_r$ ($r \geq 2$) and each $p_i - 1$ divides $n - 1$, then $n$ is a Carmichael number.
(c) Find all Carmichael numbers of the form $65p$ where $p$ is prime.

4. [20pts] (a) Determine the number of solutions of the congruence $x^{20} \equiv 13 \pmod{17}$. What is the number of solutions of $x^{20} \equiv 13 \pmod{51}$?
(b) Let $a$ and $k$ be integers greater than 1. Determine the order of $a$ modulo $(a^k - 1)$. Is it true that $k|\phi (a^k - 1)$? Justify.
(c) State Wilson’s theorem and prove that if $p$ is prime and $k$ is an integer such that $1 \leq k < p$, then $(p - k)! (k - 1)! \equiv (-1)^k \pmod{p}$ and that $(2p - 1)! \equiv p \pmod{p^2}$

5. [20pts] (a) State and prove Möbius inversion formula.
(b) Let $f(n)$ be an arithmetic function. Show that $\sum_{d\mid n} f(d) = n$ for all $n \in \mathbb{N}$ if and only if $f(n) = \phi(n)$
(c) Show that for all $n \in \mathbb{N}$, $\sum_{d\mid n} \mu(d) \phi(d) = (-1)^{\omega(n)} \prod_{p\mid n} (p - 2)$. 