King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 470  Major Exam 1
The Second Semester of 2012-2013 (122)
Time Allowed: 120mn

Name:  
ID number:  

Textbooks are not authorized in this exam

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Problem 1: Consider the first order partial differential equation

\[ u_x + e^x u_y = 1. \]  \hspace{1cm} (1)

1.) Write down the characteristic equation, and determine an explicit expression for the characteristic curve in the \(x-y\)-plane.
2.) Apply a transformation induced by the characteristics, and try to find a general solution of the transformed equation. After transforming that result back to the original variables, verify explicitly that your solution satisfies equation (1).

Problem 2: Consider the first order quasi-linear partial differential equation

\[ u_x + e^x u_y = 1. \]  \hspace{1cm} (2)

1.) Determine the characteristics of (1), here meant as curves in \(x-y-u\)-space.
2.) Solve the equation (2) around the \(x\)-axis, for the Cauchy data \(u(x, 0) = 1\) given on the \(x\)-axis.
3.) Find two different solutions of equation (2) for Cauchy data \(u(x, e^x) = x\) on the curve \(y = e^x\).
4.) Show that there is no solution of equation (2) for Cauchy data \(u(x, e^x) = 1\) (here it is sufficient to use the expression for the Cauchy data and the general solution of (2) found in Problem 1).

Problem 3: Consider the linear second order PDE with parameter \(\alpha\)

\[ u_{xx} + \alpha u_{xy} + 7u_{yy} = 0. \]  \hspace{1cm} (3)

1.) Depending on the value of \(\alpha\), determine the type of the PDE (hyperbolic, parabolic, or elliptic).
2.) For \(\alpha = -8\),
   a.) State the characteristics equations of (3).
   b.) Solve these for the characteristics curves, and sketch some of the characteristics.
   c.) Apply a change of variables and transform the PDE (3) to its canonical form.
   d.) Find a general solution of the transformed equation.
   e.) Transform the solution back to the original variables \(x\) and \(y\).
   f.) Verify that the general solution you’ve found satisfies the original equation (3).

Problem 4: Consider the linear second order PDE with parameter \(\alpha\)

\[ u_{xx} + \sqrt{28} u_{xy} + 7u_{yy} = 0. \]  \hspace{1cm} (4)

1.) State the characteristics equations of (4).
2.) Solve these for the characteristics curves.
3.) Apply a change of variables and transform the PDE (4) to its canonical form.
4.) Find a general solution of the transformed equation.
5.) Transform the solution back to the original variables \(x\) and \(y\).
6.) Verify that the general solution you’ve found satisfies the original equation (4).

Problem 5: Consider the Cauchy problem for second order PDE

\[ u_{xx} - 4u_{xy} + yu_{yy} + u_x + yu_y = 0 \]  \hspace{1cm} (5)

with Cauchy data \(u(0, y) = y^3\) and \(u_x(0, y) = 4y\) given on the \(y\)-axis.
1.) Verify that the \(y\)-axis is not a characteristic of (5).
2.) Use the Cauchy data and the PDE to compute the first four terms of the Taylor expansion of the solution of the Cauchy problem around the \(y\)-axis.
Exam 1 (MATH 470, Term 122)

1) \( u_x + e^x u_y = 1 \)

a) \( \frac{dy}{dx} = e^x \) is the characteristic equation. We integrate this equation and we find
\[
y = e^x + C.
\]
Let \( y = y - e^x \).

We now choose \( p = x \).

\[
J = \begin{vmatrix} 1 & 0 \\ e^x & 1 \end{vmatrix} = 1 \to e^x
\]

b) \( u_x = WP_x + W_y \eta x \)
\[
= WP_x + W_y \eta
\]
\[
= WP_x + W_y \eta
\]
\[
\Rightarrow WP_x - W_y \eta = 1
\]
\[
\Rightarrow WP_x = 1
\]
\[
\Rightarrow W(x,y) = f(y - e^x)
\]

So that \( u(x,y) = x + f(y - e^x) \), where \( f \) is an arbitrary differentiable function.

2) Method of characters

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = e^x, \quad \frac{du}{dt} = 1
\]

\[
x = t + a, \quad \frac{dy}{dx} = e^x \Rightarrow y = e^x + b
\]
\[
\Rightarrow y = e^x + b
\]

and \( \eta = t + c \)

iii) The Cauchy problem
\[
\begin{cases}
\phi(x,y) = u_t + 2u_x = 1 \\
u(x,0) = 1 \\
\phi(x,0) = 0
\end{cases}
\]

Assume that the characteristic passes through \( P(s,0,1) \) at time \( t = 0 \).

\[
\begin{cases}
x = a = s \\
y = e^a + b = 0 \Rightarrow b = -e^a \\
u = c = 1
\end{cases}
\]

Thus, the characteristic is
\[
\begin{cases}
x = t + s \Rightarrow s = x - t \\
y = e^{t+s} \Rightarrow y = e^{t-e^s} \\
u = t + 1 \Rightarrow t = u - 1
\end{cases}
\]

\[
\Rightarrow y = e^{t-e^s}
\]

and \( y = x - u + 1 \)

\[
x - u + 1 = e - y
\]

\[
w(x,y) = x + 1 - \ln(e - y), \quad y < e
\]
(iii) Observe that $y = e^x$ is a characteristic curve.

Let $u(x, y) = x$ and $v(x, y) = x + y - e^y$.

Both functions are solutions to $u_x + e^y u_y = 1$ and they satisfy the Cauchy data.

(iii) The general solution of the DE is

$$u(x, y) = x + f(y - e^x)$$

where $f$ is a differentiable function.

If a solution $u(x, y)$ satisfies the Cauchy data $u(x_0, y_0) = 1$, then we should find $f$ such that

$$u(x, e^x) = x + f(0) = 1$$

$$f(0) = \frac{1}{x}, \text{ for every } x.$$

Impossible.

There is no solution satisfying $u(x, e^x) = 1$.

2. $u_{xx} + u_{x} + u_{y} = 0$

a) $B^2 - 4AC = (1)^2 - 1 = \frac{28}{4}$

For $\Delta^2 < 0$ and $B = 0$, $u(x, y) = \text{Hyperbolic}$

For $\Delta^2 > 0$, $u(x, y) = \text{Elliptic}$

For $\Delta^2 = 0$, $u(x, y) = \text{Parabolic}$

b) $u_{xx} - 8u_{xy} + u_{yy} = 0$

Characteristics are

$$\frac{dy}{dx} = \frac{y - \sqrt{28}}{x} = -7$$

and

$$\frac{dy}{dx} = \frac{y + \sqrt{28}}{x} = -1$$

(iii) $\xi = y + t$, $\eta = y + x$

$$u_x = \partial_\xi \xi + \partial_\eta \eta$$

$$= t\partial_\xi + \partial_\eta$$
\[ u_{xx} = 7(w_{f_{x}}x + w_{y}y) + (w_{f_{y}}y + w_{y}y) \]
\[ = 49 w_{f_{x}} + 14 w_{f_{y}} + w_{y}y \]
\[ u_{yy} = w_{f_{y}}y + w_{y}y \]
\[ = w_{f_{y}} + w_{y} \]
\[ u_{xy} = (w_{f_{y}}y + w_{y}y) + (w_{f_{y}}y + w_{y}y) \]
\[ = w_{f_{y}} + 2w_{y} + w_{y} \]
\[ u_{xy} = 7(w_{f_{y}}y + w_{y}y) + (w_{f_{y}}y + w_{y}y) \]
\[ = 7w_{f_{y}} + 8w_{y} + w_{y} \]
\[ \Rightarrow 69w_{f_{y}} + 14w_{y} + w_{y} - 8(7w_{f_{y}} + 8w_{y} + w_{y}) = 0 \]
\[ -36w_{y} = 0 \]
\[ w_{y} = 0 \]
\[ \Rightarrow w(x,y) = f(x) + g(y) \]

And
\[ U(x,y) = f(y + x) + g(x + y) \]

where \( f, g \) are twice differentiable functions.

**Verification**
\[ u_{x} = 7f'(x + y) + g'(x + y) \]
\[ u_{xx} = 49f''(x + y) + g''(x + y) \]
\[ u_{y} = f'(y + x) + g'(x + y) \]
\[ u_{yy} = f''(x + y) + g''(x + y) \]
\[ u_{xy} = 7f''(x + y) + g''(x + y) \]
\[ \Rightarrow u_{xx} - 8u_{xy} + 7u_{yy} = 0 \]

\[ U(x,y) = \frac{\sqrt{7}}{x} = \sqrt{7} \]
\[ y = \sqrt{7}x + C \]
\[ \varepsilon = y - \sqrt{7}x, \quad \eta = x, \quad J = 1 \]

\[ u_{x} = w_{f_{x}}x + w_{y}y \]
\[ u_{xx} = 7w_{f_{x}}x + 8w_{y}y + w_{y} \]
\[ u_{yy} = w_{f_{y}}y + w_{y}y \]
\[ u_{xy} = 7w_{f_{y}}y + 8w_{y}y + w_{y} \]
\[ \Rightarrow -36w_{y} = 0 \]
\[ w_{y} = 0 \]
\[ \Rightarrow w(x,y) = f(x) + g(y) \]

\[ U(x,y) = \frac{\sqrt{7}}{x} = \sqrt{7} \]

where \( f, g \) are twice differentiable functions.

**Verification**
\[ u_{x} = f(y - \sqrt{7}x) + g'(y - \sqrt{7}x) - \sqrt{7} g'(y - \sqrt{7}x) \]
\[ u_{xx} = -2\sqrt{7} f'' + 2\sqrt{7} f'' + t - g'' \]
\[ u_{y} = x + g'(y - \sqrt{7}x) + g' \]
\[ u_{yy} = x + g'(y - \sqrt{7}x) + g'' \]
\[ u_{xy} = \frac{\sqrt{7}}{x} - \sqrt{7} g'' \]
(a) \( u_{xx} + \sqrt{3} u_{xy} + u_{yy} = 0 \)

Elliptic DE

There is no characteristic equation.

However, \( \text{det} \eta = x+y \)

\[
\frac{\eta_y}{\eta_x} = \sqrt{3} + \sqrt{3} = \frac{\sqrt{1} + \sqrt{3}}{2} = \sqrt{3} - 12 \sqrt{3}
\]

\( y = (\sqrt{3} - 12 \sqrt{3}) x + C \)

\( f = y - (\sqrt{3} - 12 \sqrt{3}) x \)

\( \mathbf{J} = \begin{pmatrix} \sqrt{3} - 12 \sqrt{3} & 1 \\ 1 & 1 \end{pmatrix} \neq 0 \)

\( \omega_x = \omega_y + \omega y \eta_y \)

\( = (\sqrt{3} - 12 \sqrt{3}) \omega_y + \omega y \eta_y \)

\( u_{xx} = -(\sqrt{3} - 12 \sqrt{3}) \omega_y + \omega_y \eta_y \)

\( = (\sqrt{3} - 12 \sqrt{3}) \omega_y - 2(\sqrt{3} - 12 \sqrt{3}) \omega y + \omega y \eta_y \)

\( \omega_y = \omega_y \eta_y + \omega y \eta_y \)

\( = \omega y + \omega y \eta_y \)

\( \omega_y = \omega y + \omega y \eta_y \)

\( \omega_{xy} = -(\sqrt{3} - 12 \sqrt{3}) \omega_y + \omega_y \eta_y \)

\( = -(\sqrt{3} - 12 \sqrt{3}) \omega y - (24 - 12 \sqrt{3}) \omega y + \omega y \eta_y \)

\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

\( \omega_y = \omega y + \omega y \eta_y \)

\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

\( \omega_{y} = \omega_y + \omega y \eta_y \)

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\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

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\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

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\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

\( \omega_{y} = \omega_y + \omega y \eta_y \)

\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

\( \omega_{y} = \omega_y + \omega y \eta_y \)

\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)

\( \omega_{y} = \omega_y + \omega y \eta_y \)

\( \omega_{xy} = \frac{2}{\sqrt{3} - 12 \sqrt{3}} \omega_y - 2(\sqrt{12} - 12 \sqrt{12}) \omega y + \omega y \eta_y \)
3.1 a) The characteristic equation are

\[
\frac{dy}{dx} = -2 \pm \sqrt{4-4y}
\]

The \(y\)-axis bisects equation \(x = 0\).

Any point \(y_0\) on \(x\) has a vertical tangent.

But, any point \(y_0\) on \(y\) a characteristic curve has a tangent of slope \(\frac{dy}{dx} = -2 \pm \sqrt{4-4y_0}\),

which excludes vertical tangents.

Consequently, \(x = 0\) is not a characteristic curve in the equation.