King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 470 Major Exam 1
The Second Semester of 2012-2013 (122)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		18
2		23
3		23
4		18
5		18
Total		100

Problem 1: Consider the first order partial differential equation

$$u_x + e^x u_y = 1. \tag{1}$$

1.) Write down the characteristic equation, and determine an explicit expression for the characteristic curve in the x-y-plane.

2.) Apply a transformation induced by the characteristics, and try to find a general solution of the transformed equation. After transforming that result back to the original variables, verify explicitly that your solution satisfies equation (1).

Problem 2: Consider the first order quasi-linear partial differential equation

$$u_x + e^x u_y = 1. \tag{2}$$

1.) Determine the characteristics of (1), here meant as curves in x-y-u-space.

2.) Solve the equation (2) around the x-axis, for the Cauchy data $u(x, 0) = 1$ given on the x-axis.

3.) Find two different solutions of equation (2) for Cauchy data $u(x, e^x) = x$ on the curve $y = e^x$.

4.) Show that there is no solution of equation (2) for Cauchy data $u(x, e^x) = 1$ (here it is sufficient to use the expression for the Cauchy data and the general solution of (2) found in Problem 1).

Problem 3: Consider the linear second order PDE with parameter α

$$u_{xx} + \alpha u_{xy} + 7u_{yy} = 0. \tag{3}$$

1.) Depending on the value of α, determine the type of the PDE (hyperbolic, parabolic, or elliptic).

2.) For $\alpha = -8$,

a.) State the characteristics equations of (3).

b.) Solve these for the characteristics curves, and sketch some of the characteristics.

c.) Apply a change of variables and transform the PDE (3) to its canonical form.

d.) Find a general solution of the transformed equation.

e.) Transform the solution back to the original variables x and y.

f.) Verify that the general solution you've found satisfies the original equation (3).

Problem 4: Consider the linear second order PDE with parameter α

$$u_{xx} + \sqrt{28}u_{xy} + 7u_{yy} = 0. \tag{4}$$

1.) State the characteristics equations of (4).

2.) Solve these for the characteristics curves.

3.) Apply a change of variables and transform the PDE (4) to its canonical form.

4.) Find a general solution of the transformed equation.

5.) Transform the solution back to the original variables x and y.

6.) Verify that the general solution you've found satisfies the original equation (4).

Problem 5: Consider the Cauchy problem for second order PDE

$$u_{xx} - 4u_{xy} + yu_{yy} + u_x + yu_y = 0 \tag{5}$$

with Cauchy data $u(0, y) = y^3$ and $u_x(0, y) = 4y$ given on the y-axis.

1.) Verify that the y-axis is not a characteristic of (5).

2.) Use the Cauchy data and the PDE to compute the first four terms of the Taylor expansion of the solution of the Cauchy problem around the y-axis.

Exam I (MATH 470, term 122)

1) $u_x + e^x u_y = 1$

a) $\frac{dy}{dx} = e^x$ is the characteristic equation. We integrate this equation, and we find

$y = e^x + C$.

let $\eta = y - e^x$.

We now choose $\xi = x$.

$J = \begin{vmatrix} 1 & 0 \\ -e^x & 1 \end{vmatrix} = 1 \neq 0$

b) $u_x = w_\xi \xi_x + w_\eta \eta_x$

$= w_\xi - e^x w_\eta$

$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\eta$

$\Rightarrow w_\xi - e^x w_\eta + e^x (w_\eta) = 1$

$\Rightarrow w_\xi = 1$

$\Rightarrow w(\xi, \eta) = \xi + f(\eta)$

So that, $\boxed{u(x,y) = x + f(y - e^x)}$, where f is an arbitrary differentiable function.

$u_x = 1 - e^x f'(y - e^x)$

$u_y = f'(y - e^x)$

$u_x + e^x u_y = 1 - e^x f'(y - e^x) + e^x f'(y - e^x)$

$= 1$.

$\Rightarrow u(x,y)$ is a solution to the initial PDE.

b) Method of characteristics

i) $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = e^x$, $\frac{du}{dt} = 1$

$x = t + a$, $\frac{dy}{dx} = e^x \Rightarrow y = e^x + b$

$\Rightarrow y = e^{t+a} + b$

and $u = t + c$

ii) The Cauchy problem

$$\begin{cases} u_x + e^x u_y = 1 \\ u(x,0) = 1 \quad \text{on } \Gamma: y = 0 \end{cases}$$

Assume that the characteristic passes through $P(s, 0, 1)$ at time $t = 0$.

$$\begin{cases} x = a = s \\ y = e^a + b = 0 \Rightarrow b = -e^s \\ u = c = 1 \end{cases}$$

Thus, the characteristic are

$$\begin{cases} x = t + s & \rightarrow s = x - t \\ y = e^{t+s} - e^s & \rightarrow y = e^s(e^t - 1) \\ u = t + 1 & \rightarrow t = u - 1 \end{cases}$$

$\Rightarrow y = e^{x-t}(e^t - 1)$

and $y = e^{x-u+1}(e^{u-1} - 1) = e^x - e^{x-u+1}$

$e^{x-u+1} = e^x - y$

$$\boxed{u(x,y) = x + 1 - \ln(e^x - y), \quad y < e^x}$$

iii) Observe that

$y = e^x$ is a characteristic curve.

Let $u(x,y) = x$ and

$u(x,y) = x + y - e^x$.

Both functions are solutions to $u_x + e^x u_y = 1$ and they satisfy the Cauchy data.

iv) The general solution of the DE is

$u(x,y) = x + f(y - e^x)$, where

f is a differentiable function.

If a solution $u(x,y)$ satisfies the Cauchy data $u(x, e^x) = 1$, then, we should find f such that

$u(x, e^x) = x + f(e^x - e^x) = 1$

$x + f(0) = 1$

$f(0) = \frac{1}{x}$, for every x.

<u>Impossible</u>.

There is no solution satisfying $u(x, e^x) = 1$.

2.) $u_{xx} + \alpha u_{xy} + 7 u_{yy} = 0$

a) $B^2 - AC = \left(\frac{\alpha}{2}\right)^2 - 7 = \frac{\alpha^2 - 28}{4}$

for $\alpha \in (-\infty, -\sqrt{28}) \cup (\sqrt{28}, \infty)$: Hyperbolic

for $\alpha \in (-\sqrt{28}, \sqrt{28})$: Elliptic

for $\alpha = -\sqrt{28}, \sqrt{28}$: Parabolic

b) $u_{xx} - 8 u_{xy} + 7 u_{xy} = 0$

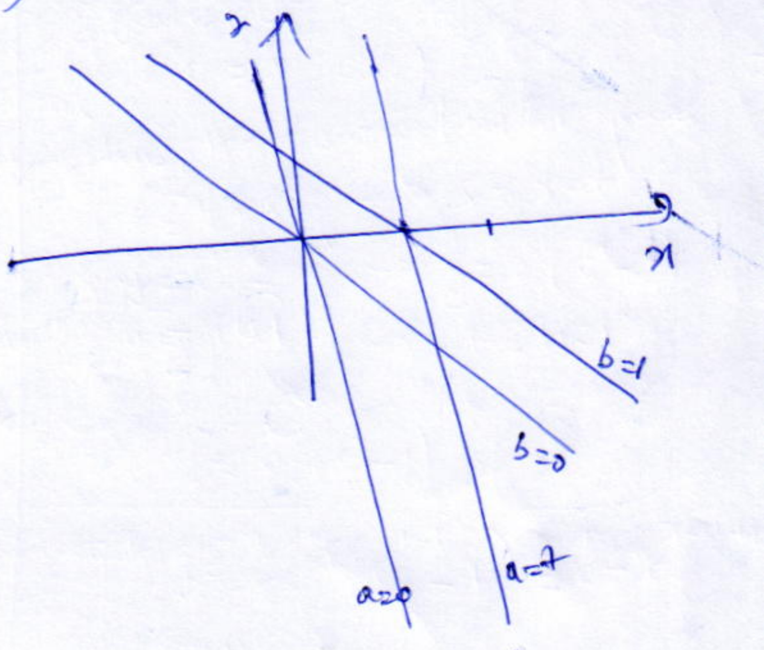
i) characteristics are

$$\frac{dy}{dx} = \frac{-4 - \sqrt{\frac{36}{4}}}{1} = -7$$

and

$$\frac{dy}{dx} = \frac{-4 +}{1} = -1$$

ii) $y = -7x + a$ and $y = -x + b$



(iii) $\xi = y + 7x$, $\eta = y + x$.

$u_x = w_\xi \xi_x + w_\eta \eta_x$

$= 7 w_\xi + w_\eta$