

King Fahd University of Petroleum and Minerals  
Department of Mathematics & Statistics  
Math 470 Major Exam 2  
The Second Semester of 2012-2013 (122)

Time Allowed: 120mn

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Name:

ID number:

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Textbooks are not authorized in this exam

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Problem #	Marks	Maximum Marks
1		22
2		22
3		20
4		18
5		18
Total		100

**Problem 1:** Suppose that an infinite string has initial displacement

$$u(x, 0) = f(x) = \begin{cases} x + 1, & -1 \leq x \leq 0 \\ 1 - 2x, & 0 \leq x \leq 1/2 \\ 0, & x < -1 \text{ and } x > 1/2 \end{cases}$$

and zero initial velocity  $u_t(x, 0) = 0$ . Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs  $u(x, 0) = f(x)$  and  $u_t(x, 0) = 0$  using D'Alembert's formula. Illustrate the nature of the solution by sketching the  $ux$ -profiles  $y = u(x, t)$  of the string displacement for  $t = 0$ ,  $t = 1/2$  and  $t = 1$ .

**Problem 2:** The acoustic pressure in an organ pipe obeys the 1-D wave equation

$$p_{tt} = c^2 p_{xx}$$

where  $c$  is the speed of sound in air. Each organ pipe is closed at one end and open at the other. At the closed end, the BC is that  $p_x(0, t) = 0$ , while at the open end, the BC is  $p(L, t) = 0$ , where  $L$  is the length of the pipe. Given initial conditions  $p(x, 0) = f(x)$  and  $p_t(x, 0) = g(x)$ , use separation of variables to determine the Fourier-series solution of the IBVP.

**Problem 3:** Solve the IBVP

$$\begin{aligned} u_{tt} &= 9u_{xx} + 54x^2 \quad \text{for } 0 < x < 1, \quad t > 0 \\ u(x, 0) &= u_t(x, 0) = 0 \quad \text{for } 0 \leq x \leq 1 \\ u(0, t) &= u(1, t) = 0 \quad \text{for } t \geq 0. \end{aligned}$$

(Hint: write  $u(x, t) = U(x, t) + f(x)$  in which  $U$  satisfies an homogeneous PDE and solve this PDE to find  $U$ , then deduce the solution  $u$ )

**Problem 4:** Consider the homogeneous heat problem

$$u_t = u_{xx}; \quad u_x(0, t) = 0 = u_x(1, t); \quad u(x, 0) = f(x)$$

where  $t > 0$ ,  $0 \leq x \leq 1$  and  $f$  is a piecewise smooth function on  $[0, 1]$ .

- 1.) Determine the Fourier-solution of the IBVP.
- 2.) Write explicitly the solution in the case  $f(x) = u_0$ ,  $u_0$  is a constant.

**Problem 5:** Consider the diffusion problem on the positive spatial axis

$$u_t = ku_{xx}, \quad \text{for } x > 0, \quad t > 0,$$

with initial and boundary conditions

$$u(x, 0) = f(x) \quad \text{for } x \geq 0, \quad u_x(0, t) = 0 \quad \text{for } t \geq 0.$$

Here  $k$  is a positive constant,  $f$  a given function. Apply separation of variables to obtain a bounded solution of the problem in terms of Fourier integral.

# MATH 47D (122)

## Exam 2

$$1.) u(x,t) = \frac{1}{2} [f(x-t) + f(x+t)]$$

When  $t=0$ ,

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \leq x \leq 0 \\ 1-2x, & 0 \leq x \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

When  $t = \frac{1}{2}$

$$u(x, \frac{1}{2}) = \frac{1}{2} [f(x-\frac{1}{2}) + f(x+\frac{1}{2})]$$

$$f(x-\frac{1}{2}) = \begin{cases} x-\frac{1}{2}+1, & -1 \leq x-\frac{1}{2} \leq 0 \\ 1-2(x-\frac{1}{2}), & 0 \leq x-\frac{1}{2} \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

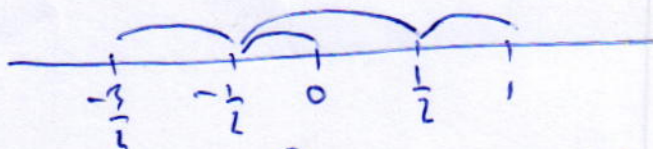
$$f(x+\frac{1}{2}) = \begin{cases} x+\frac{1}{2}+1, & -1 \leq x+\frac{1}{2} \leq 0 \\ 1-2(x+\frac{1}{2}), & 0 \leq x+\frac{1}{2} \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$-1 \leq x-\frac{1}{2} \leq 0 \Leftrightarrow -\frac{1}{2} \leq x \leq \frac{1}{2}$$

$$0 \leq x-\frac{1}{2} \leq \frac{1}{2} \Leftrightarrow \frac{1}{2} \leq x \leq 1$$

$$-1 \leq x+\frac{1}{2} \leq 0 \Leftrightarrow -\frac{3}{2} \leq x \leq -\frac{1}{2}$$

$$0 \leq x+\frac{1}{2} \leq \frac{1}{2} \Leftrightarrow -\frac{1}{2} \leq x \leq 0$$



$$\Rightarrow u(x, \frac{1}{2}) = \begin{cases} \frac{1}{2}(x+\frac{3}{2}), & -\frac{3}{2} \leq x \leq -\frac{1}{2} \\ \frac{1}{2}(\frac{1}{2}-x), & -\frac{1}{2} \leq x \leq 0 \\ \frac{1}{2}(x+\frac{1}{2}), & 0 \leq x \leq \frac{1}{2} \\ 1-x, & \frac{1}{2} \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

When  $t=1$

$$u(x,1) = \frac{1}{2} [f(x-1) + f(x+1)]$$

$$f(x-1) = \begin{cases} x-1+1, & -1 \leq x-1 \leq 0 \\ 1-2(x-1), & 0 \leq x-1 \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

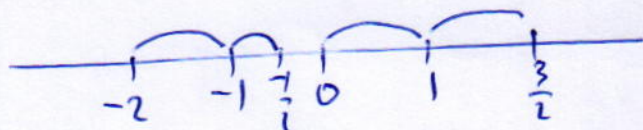
$$f(x+1) = \begin{cases} x+1+1, & -1 \leq x+1 \leq 0 \\ 1-2(x+1), & 0 \leq x+1 \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$-1 \leq x-1 \leq 0 \Leftrightarrow 0 \leq x \leq 1$$

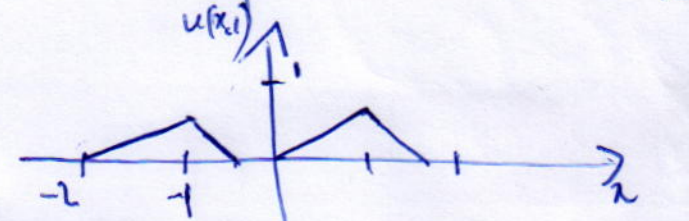
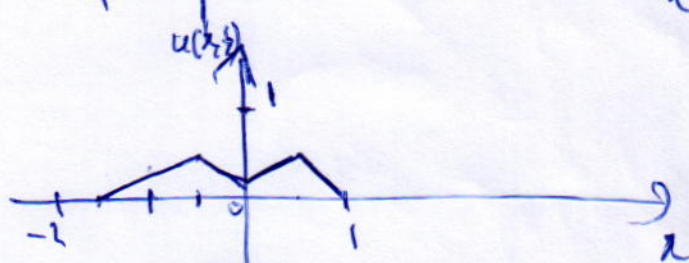
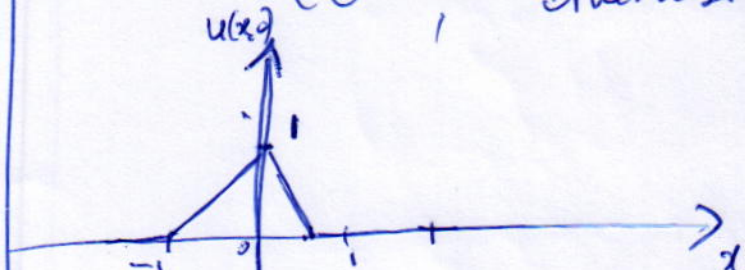
$$0 \leq x-1 \leq \frac{1}{2} \Leftrightarrow 1 \leq x \leq \frac{3}{2}$$

$$-1 \leq x+1 \leq 0 \Leftrightarrow -2 \leq x \leq -1$$

$$0 \leq x+1 \leq \frac{1}{2} \Leftrightarrow -1 \leq x \leq -\frac{1}{2}$$



$$u(x,1) = \begin{cases} \frac{1}{2}(x+2), & -2 \leq x \leq -1 \\ -\frac{1}{2}(1+2x), & -1 \leq x \leq -\frac{1}{2} \\ \frac{x}{2}, & 0 \leq x \leq 1 \\ \frac{1}{2}(3-2x), & 1 \leq x \leq \frac{3}{2} \\ 0, & \text{otherwise} \end{cases}$$



12.

$$\begin{cases} P_{tt} = c^2 P_{xx}, & 0 < x < L, t > 0 \\ P_x(0, t) = 0 = P_x(L, t) = 0 \\ P(x, 0) = f(x), P_t(x, 0) = g(x) \end{cases}$$

Let  $P(x, t) = X(x)T(t)$

$$X T'' = c^2 X'' T \Leftrightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

$$\Leftrightarrow \begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases} \quad (1)$$

$$T'' + c^2 \lambda T = 0$$

We solve (1)

$\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = C_1 + C_2 x$   
 $X'(x) = C_2, X'(0) = 0 \Rightarrow C_2 = 0$

$$X(L) = 0 \Rightarrow C_1 = 0$$

Thus,  $P = 0$

$\lambda < 0, \lambda = -\alpha^2$

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

$$X'(x) = C_1 \alpha \sinh \alpha x + C_2 \alpha \cosh \alpha x$$

$$X'(0) = 0 \Rightarrow C_2 = 0$$

$$X(L) = 0 \Rightarrow C_1 \cosh \alpha L = 0$$

$$\Rightarrow C_1 = 0$$

Thus,  $P = 0$

$\lambda > 0, \lambda = \alpha^2$

$$X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$$

$$X'(x) = -C_1 \alpha \sin \alpha x + C_2 \alpha \cos \alpha x$$

$$X'(0) = 0 \Rightarrow C_2 = 0$$

$$X(L) = 0 \Rightarrow C_1 \cos \alpha L = 0$$

$$\cos \alpha L = 0$$

$$\alpha L = \frac{\pi}{2} + n\pi$$

$$\Rightarrow \alpha_n = \left(\frac{1}{2} + n\right) \frac{\pi}{L} = \frac{(2n+1)\pi}{2L}$$

$$n = 0, 1, 2, \dots$$

$$X_n(x) = C_n \cos\left(\frac{(2n+1)\pi}{2L} x\right)$$

$$T'' + c^2 \alpha_n^2 T = 0$$

$$T_n(t) = C_1 \cos(c \alpha_n t) + C_2 \sin(c \alpha_n t)$$

$$\Rightarrow P(x, t) = \sum_{n=0}^{\infty} \left( A_n \cos(c \alpha_n t) + B_n \sin(c \alpha_n t) \right) \cos \alpha_n x$$

$$P_t(x, 0) = \sum_{n=0}^{\infty} c \alpha_n \left( -A_n \sin c \alpha_n t + B_n \cos c \alpha_n t \right) \cos \alpha_n x$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} A_n \cos \alpha_n x$$

$$\Rightarrow A_n = \frac{\int_0^L f(x) \cos\left(\frac{(2n+1)\pi}{2L} x\right) dx}{\int_0^L \cos^2\left(\frac{(2n+1)\pi}{2L} x\right) dx}$$

$$= \frac{2}{L} \int_0^L f(x) \cos\left(\frac{(2n+1)\pi}{2L} x\right) dx$$

$$P_t(x, 0) = g(x) = \sum_{n=0}^{\infty} c \alpha_n B_n \cos \alpha_n x$$

$$c \alpha_n B_n = \frac{2}{L} \int_0^L g(x) \cos \alpha_n x dx$$

$$B_n = \frac{2}{c \alpha_n} \int_0^L g(x) \cos \alpha_n x dx$$