King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 470 Final Exam
The Second Semester of 2012-2013 (122)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		25
3		25
4		30
Total		100

Problem 1: Consider the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial D \end{aligned}$$

where D is a simply-connected 2D region with piecewise smooth boundary ∂D.
1.) State the Maximum Principle for u on D. If $f = 10$ at each point on the boundary ∂D, what is u on D? Explain your answer.
2.) Now let D be the disc of radius R centered at the origin,

$$D = \{(x, y) : \ x^2 + y^2 \leq R\}.$$

Name and state (without proof) another property of u which gives the value of u at the center of the disc in terms of the values of u on the boundary $\partial D = \{(x, y) : \ x^2 + y^2 = R\}$. Use this result to find $u(0, 0)$ if on the boundary u takes the values

$$u(R, \theta) = \begin{cases} 90, & -\pi/2 \leq \theta \leq \pi/2, \\ 25, & \pi/2 \leq \theta \leq \pi, \\ 7, & \pi \leq \theta \leq 3\pi/2. \end{cases}$$

Problem 2: Solve the Laplace equation in the rectangle $0 < x < a$, $0 < y < b$,

$$\nabla^2 v(x, y) = 0,$$

with boundary conditions

$$\begin{aligned} v(0, y) = v(a, y) &= v(x, b) = 0, \\ v(x, 0) &= \cos\left(5\frac{\pi}{a}x\right). \end{aligned}$$

Problem 3: 1.) Use Fourier integral or Fourier transform method to prove that the solution of the Laplace equation for the lower half-plane, whose boundary conditions is the horizontal axis:

$$\begin{aligned} \nabla^2 u(x, y) &= 0, \quad \text{for } -\infty < x < \infty, \ y < 0, \\ u(x, 0) &= f(x) \quad \text{for } -\infty < x < \infty, \end{aligned}$$

is

$$u(x, y) = -\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2} d\xi.$$

2.) Write the solution for

$$f(x) = \begin{cases} 0, & |x| > 2, \\ x^2, & -2 \leq x \leq 2. \end{cases}$$

Problem 4: 1.) Solve the Laplace equation on the quarter unit disc

$$\nabla^2 v(r, \theta) = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0,$$

with boundary conditions

$$v(1,\theta) = g(\theta), \quad v(0,\theta) \text{ bounded}, \quad 0 < \theta < \pi/2,$$
$$v(r,0) = 0, \quad v(r,\frac{\pi}{2}) = 0, \quad 0 < r < 1.$$

2.) Solve the heat equation problem on the unit quarter disc

$$u_t = \nabla^2 u(r,\theta,t), \quad 0 < r < 1, \quad 0 < \theta < \pi/2, \quad t > 0,$$

with boundary conditions

$$u(1,\theta,t) = g(\theta), \quad u(0,\theta,t) \text{ bounded}, \quad 0 < \theta < \pi/2, \quad t > 0,$$
$$u(r,0,t) = 0, \quad u(r,\frac{\pi}{2},t) = 0, \quad 0 < r < 1, \quad t > 0,$$

with initial condition

$$u(r,\theta,0) = f(r,\theta), \quad 0 < r < 1, \quad 0 < \theta < \pi/2.$$

Do not evaluate the coefficients in the solution. (Hint: set $w(r,\theta,t) = u(r,\theta,t) - v(r,\theta)$, and solve the problem for w).

3.) Prove the solution to 2.) is unique. (Hint: write the equation for the difference $h = u_1 - u_2$ of two solutions u_1 and u_2, multiply this equation by h, and apply the Divergence Theorem, and do not integrate by parts. No need to use r and θ, just denote the region by D.)

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Q1:

1) u achieves its maximum and minimum values on $\bar{D}$ only at points of ∂D.

Let $v = u - 10$.

Thus $\nabla^2 v = 0$ in D
$v = 0$ on ∂D

From the Maximum principle,
$v = 0$ in $\bar{D}$
and $u = 10$ on D

2) Mean value property:

$$u(x) = \frac{1}{2\pi\varepsilon} \oint_{C} u(y)\, d\sigma_y, \text{ for } x \in D$$

C is a circle of radius ε about x such that interior $C \subset D$.

Here $\varepsilon = R$, $x = (0,0)$, $D = \text{disc}(0,R)$

$$u(0,0) = \frac{1}{2\pi R} \oint_{C} u(R,\theta) R\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(\theta)\, d\theta$$

$$= \frac{1}{2\pi}\left(90\pi + 25\frac{\pi}{2} + 7\frac{\pi}{2}\right)$$

$$= \frac{53}{2}$$

Q2:

$$\begin{cases} \nabla^2 v(x,y) = 0 \\ v(0,y) = v(a,y) = v(x,b) = 0 \\ v(x,0) = \cos\left(5\frac{\pi}{a}x\right) \end{cases}$$

Use seperation of variables.

$v = XY$

$X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(a) = 0 \end{cases}, \quad \begin{cases} Y'' - \lambda Y = 0 \\ Y(b) = 0 \end{cases}$$

$\lambda = \alpha^2 \Rightarrow X = C_1 \cos \alpha x + B_2 \sin \alpha x$

$X(0) = 0 \Rightarrow C_1 = 0$

$X(a) = 0 \Rightarrow \sin \alpha a = 0$, $\alpha_n a = n\pi$

$\alpha_n = \frac{n\pi}{a}$, $n = 1, 2, 3 \ldots$

$Y'' - \alpha_n^2 Y = 0$

$Y = C_1 e^{\alpha_n y} + C_2 e^{-\alpha_n y}$

$Y(b) = 0 \Rightarrow C_1 e^{\alpha_n b} + C_2 e^{-\alpha_n b} = 0$

$C_1 = -C_2 e^{-2\alpha_n b}$

$$\Rightarrow Y_n = -C_2 e^{-2\alpha_n b} e^{\alpha_n y} + C_2 e^{-\alpha_n y}$$

$$= -C_2 e^{-\alpha_n b}\left(e^{-\alpha_n b} e^{\alpha_n y} - e^{\alpha_n b} e^{-\alpha_n y}\right)$$

$$= -C_2 e^{-\alpha_n b}\left(e^{+\alpha_n (y-b)} - e^{-\alpha_n (y-b)}\right)$$

$$= C_n \sinh \alpha_n (y-b)$$

$$\Rightarrow v(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a}(y-b)$$