MATH 470.1 (Term 122)

Homework Exercises 1 (Sects. 1.1-1.4, 2.1-2.4) Due date: Feb. 18, 2013

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1. Consider the linear first order PDE problem

$$u_t + 2xu_x = -u, \quad t > 0, \quad -\infty < x < \infty. \tag{1}$$

a.) Write down the characteristic equation of (1), and determine a parameterized explicit expression for the characteristic lines. "Characteristics" are meant here as curves in the t-x-plane.
b.) By applying a transformation induced by the characteristics, find a general solution of (1) that includes an arbitrary smooth function f. Show explicitly that your solution satisfies equation (1).

2. Investigate the first order quasi-linear PDE

$$u_x + 3u_y = u^2 \tag{2}$$

by means of the *method of characteristics*.
a.) Determine the characteristics of (2), here meant as curves in x-y-u-space.
b.) Using the result of a.), solve the equation (2) for Cauchy data $u(x, -3x) = 1/3$ given on the straight curve $y = -3x$.
c.) Find two solutions of equation (2) for Cauchy data $u(x, 3x) = -1/x$ on the line $y = 3x$.

3. Look at a general second order linear PDE of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \tag{3}$$

where A, B, C, D, E, F and G are given functions of x, y. Consider a transformation to new independent variables ξ, η, such that the transformed function w relates to the original unknown u as $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$.
a.) Show that the transformation leads to an equation for w of the form

$$aw_{\xi\xi} + 2bw_{\xi\eta} + cw_{\eta\eta} + dw_\xi + ew_\eta + fw + g = 0. \tag{4}$$

Determine the transformed coefficients a, b, c, d, e, f and g (functions of ξ, η).
b.) Let $J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$ be the Jacobian of the transformation. Show that the coefficients of the transformed (4) and of the original equation (3) satisfy the relation

$$b^2 - ac = (B^2 - AC)J^2. \tag{5}$$

4. Consider the following second order linear PDEs. In each case, classify the equation as hyperbolic, parabolic, or elliptic, state and solve the characteristics equations, and, depending on the respective type, apply a transformation that leads to the canonical form of the PDE. If possible, find a general solution that involves two arbitrary functions, and show explicitly that your solution, after back transformation, satisfies the original PDE.
a.) $-2u_{xx} + u_{xy} + u_{yy} = 0$,
b.) $\frac{5}{2}u_{xx} + u_{xy} + u_{yy} = 0$,
c.) $9u_{xx} + 12u_{xy} + 4u_{yy} = 0$.

Homework 1

1.) $u_t + 2x\,u_x = -u$ ①

This is a linear 1^{st} order PDE

a) The characteristic equation is

$$\frac{dx}{dt} = 2x \iff x = ce^{2t} \text{ or } \ln x - 2t = C,\ x>0$$

b) Let $\eta = \ln x - 2t$, $\xi = t$

$$J = \begin{vmatrix} 1 & 0 \\ -2 & \frac{1}{x} \end{vmatrix} = \frac{1}{x} \neq 0$$

Let $u(x,t) = w(\xi,\eta)$

$$u_t = w_\xi \xi_t + w_\eta \eta_t = w_\xi - 2w_\eta$$

$$u_x = w_\xi \xi_x + w_\eta \eta_x = \frac{1}{x} w_\eta$$

When substitute in ①, we find

$$w_\xi = -w \quad ②$$

We integrate ②, and we find

$$w(\xi,\eta) = f(\eta)\,e^{-\xi}$$

Therefore,

$$\boxed{u(x,t) = e^{-t} f(\ln x - 2t)}$$, where f is a differentiable function.

$$u_t = \left[-f(\ln x - 2t) - 2f'(\ln x - 2t)\right]e^{-t}$$

$$u_x = \frac{1}{x} e^{-t} f'(\ln x - 2t)$$

$$\Rightarrow u_t + 2x\,u_x = -f(\ln x - 2t)\,e^{-t} = -u$$

2.) $u_x + 3u_y = u^2$

a quasi-linear PDE of 1^{st} order.

Characteristic equations

$$\frac{dx}{dt} = 1,\quad \frac{dy}{dt} = 3,\quad \frac{du}{dt} = u^2$$

$$x = t + a,\quad y = 3t + b,\quad u = \frac{1}{-t+c}$$

b) $\Gamma: y = -3x$, $u(x,-3x) = 1/3$.

We parametrize Γ as

$$x = s,\quad y = -3s,\quad u(s,-3s) = \frac{1}{3}$$

We assume that a characteristic passes through $P(s,-3s,\frac{1}{3})$ at $t=0$.

So, we have

$$a = s,\quad b = -3s \quad \frac{1}{c} = \frac{1}{3}$$

$$\Rightarrow x = t + s,\quad y = 3t - 3s,\quad u = \frac{1}{3-t}$$

$$\Rightarrow s = x - t,\quad s = t - \frac{1}{3}y,\quad t = 3 - \frac{1}{u}$$

So, $x - t = t - \frac{1}{3}y$

$$x + \frac{1}{3}y = 2t = 2\left(3 - \frac{1}{u}\right)$$

$$x + \frac{1}{3}y + \frac{2}{u} = 6$$

$$\boxed{u(x,y) = \frac{2}{6 - x - \frac{y}{3}}}$$

c) $\Gamma: y = 3x$ is a characteristic curve

$$u(x,y) = -\frac{1}{x} \quad \text{and} \quad u(x,y) = -\frac{3}{y}$$

are two solutions such that

$$u(x,3x) = -\frac{1}{x}.$$

3.) $A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0$

Let ξ, η such that $J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$, for $(x,y) \in D$.

$u_x = w_\xi \xi_x + w_\eta \eta_x$

$u_y = w_\xi \xi_y + w_\eta \eta_y$

$$u_{xx} = (w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x)\xi_x + w_\xi \xi_{xx} + (w_{\xi\eta}\xi_x + w_{\eta\eta}\eta_x)\eta_x + w_\eta \eta_{xx}$$

$$= w_{\xi\xi}\xi_x^2 + 2 w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx}$$

$$u_{yy} = (w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y)\xi_y + w_\xi \xi_{yy} + (w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y)\eta_y + w_\eta \eta_{yy}$$

$$= w_{\xi\xi}\xi_y^2 + 2 w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy}$$

$$u_{xy} = (w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y)\xi_x + w_\xi \xi_{xy} + (w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y)\eta_x + w_\eta \eta_{xy}$$

$$= w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \eta_x\xi_y) + w_{\eta\eta}\eta_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}$$

Now, we substitute in the original equation.

We find,

$$(A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2)w_{\xi\xi} + (2A\xi_x\eta_x + 2B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y)w_{\xi\eta} + (A\eta_x^2 + 2B\eta_y\eta_x + C\eta_y^2)w_{\eta\eta}$$
$$+ (A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y)w_\xi$$
$$+ (A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y)w_\eta$$
$$+ Fw + G = 0$$

$$\begin{cases} a = A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 \\ b = A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\eta_y\xi_y \\ c = A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2 \\ d = A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ e = A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ f = F, \quad g = G \end{cases}$$

b.)

$$b^2 - ac = A^2\xi_x^2\eta_x^2 + B^2(\xi_x\eta_y + \xi_y\eta_x)^2 + C^2\eta_y^2\xi_y^2 + 2AB\xi_x\eta_x(\xi_x\eta_y + \xi_y\eta_x)$$
$$+ 2AC\xi_x\eta_x\xi_y\eta_y + 2BC\xi_y\eta_y(\xi_x\eta_y + \xi_y\eta_x)$$
$$- (A^2\xi_x^2\eta_x^2 + 4B^2\xi_x\xi_y\eta_x\eta_y + C^2\xi_y^2\eta_y^2)$$
$$- (2AB\xi_x^2\eta_x\eta_y + AC\xi_x^2\eta_y^2 + 2AB\eta_x^2\xi_x\xi_y)$$
$$- (2BC\xi_x\xi_y\eta_y^2 + AC\xi_y^2\eta_x^2 + 2BC\xi_y^2\eta_x\eta_y)$$
$$= B^2(\xi_x\eta_y - \xi_y\eta_x)^2 - AC(\xi_x\eta_y - \xi_y\eta_x)^2$$
$$= J^2(B^2 - AC)$$