

MATH 470.1 (Term 122)
Homework Exercises 3 (Sects. 5.6-5.8) Due date: April 29, 2013

1. Assume that u satisfies an inhomogeneous 1-D wave equation on an interval $[0, L]$

$$u_{tt} = c^2 u_{xx} + f(x, t), \text{ for } 0 < x < L, t > 0, \quad (1)$$

with given inhomogeneity f and constant phase velocity $c > 0$.

a.) Show that the following integral relationship holds for any interval $[a, b]$ with $0 < a, b < L$:

$$\frac{d}{dt} \int_a^b \frac{1}{2} (u_t^2 + c^2 u_x^2) dx = c^2 u_t u_x \Big|_a^b + \int_a^b f u_t dx \quad (2)$$

(Hint: Multiply the wave equation by u_t , integrate over $[a, b]$, then use integration by parts on a suitable term. Note that $\partial_t(u_t^2) = 2u_t u_{tt}$).

b.) Show that the IBVP (1), with

$$u(0, t) = u(L, t) = 0, \quad u(x, 0) = g(x), \text{ for } 0 \leq x \leq L, t \geq 0,$$

with g a given function, can have only one solution. (Hint: Assume there are two solutions; take the difference and determine the conditions (PDE & boundary/initial values) that are satisfied by the function. Evaluate the result a.) above for the difference function).

2. Consider the following heat problem in dimensionless variables

$$u_t = u_{xx} + \frac{\pi^2}{4} u - b, \quad 0 < x < 1, \quad t > 0 \quad (3)$$

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0 \quad (4)$$

$$u(x, 0) = u_0, \quad 0 < x < 1. \quad (5)$$

a.) Derive the equilibrium solution $u_E(x)$, that is, when $u_t = 0$.

b.) Using $u_E(x)$, transform the given heat problem for $u(x, t)$ into the following problem for a function $v(x, t)$:

$$v_t = v_{xx} + \frac{\pi^2}{4} v, \quad 0 < x < 1, \quad t > 0 \quad (6)$$

$$v(0, t) = 0, \quad v(1, t) = 0, \quad t > 0 \quad (7)$$

$$v(x, 0) = f(x), \quad 0 < x < 1, \quad (8)$$

where $f(x)$ will be determined by the transformation.

c.) For an appropriate value of α show that the transformation $w(x, t) = e^{\alpha t} v(x, t)$ further simplifies the problem to

$$w_t = w_{xx}, \quad 0 < x < 1, \quad t > 0 \quad (9)$$

$$w(0, t) = 0, \quad w(1, t) = 0, \quad t > 0 \quad (10)$$

$$w(x, 0) = f(x), \quad 0 < x < 1. \quad (11)$$

d.) Deduce the solution u of (3)-(5).

3. Suppose u is a bounded solution of the initial value problem

$$u_t = Ku_{xx} \text{ for } -\infty < x < \infty, t > 0, u(x, 0) = f(x) \text{ for } -\infty < x < \infty \quad (12)$$

for a given continuous function f , with $u(x, t) \rightarrow 0$ uniformly in t as $x \rightarrow \pm\infty$. K is a positive constant. Show that

$$|u(x, t)| \leq \max_{x \in \mathbb{R}} |f(x)| \quad (13)$$

for all x, t with $-\infty < x < \infty$ and $t > 0$. (Hint: apply the weak maximum principle on an interval $x \in [-a, a]$ for $t \geq 0$, and take the limit $a \rightarrow \infty$).

Q1: $u_{tt} = c^2 u_{xx} + f(x, t)$ (1)

a) We multiply (1) by u_t

$$u_{tt} u_t = c^2 u_{xx} u_t + f(x, t) u_t \quad (2)$$

We integrate (2) on $x \in [a, b]$.

$$\int_a^b u_{tt} u_t dx = c^2 \int_a^b u_{xx} u_t dx + \int_a^b f(x, t) u_t dx$$

Now, $\int_a^b u_{tt} u_t dx = \frac{1}{2} \frac{d}{dt} \int_a^b u_t^2 dx$

We integrate by parts in $\int_a^b u_{xx} u_t dx$.

$$\begin{aligned} \int_a^b u_{xx} u_t dx &= [u_x u_t]_a^b - \int_a^b u_x u_{xt} dx \\ &= [u_x u_t]_a^b - \frac{1}{2} \frac{d}{dt} \int_a^b u_x^2 dx \end{aligned}$$

thus,

$$\frac{1}{2} \frac{d}{dt} \int_a^b (u_t^2 + c^2 u_x^2) dx = c^2 [u_x u_t]_a^b + \int_a^b f(x, t) u_t dx \quad (3)$$

b.) Assume there are two solutions u_1 and u_2 .

Let $w = u_1 - u_2$
Thus, w satisfies the IVP

$$\begin{cases} w_{tt} = c^2 w_{xx} \\ w(0, t) = w(L, t) = 0 \\ w(x, 0) = 0, \quad 0 < x < L, t > 0 \end{cases}$$

Apply the estimate (3) from previous question, we find

$$\frac{d}{dt} \int_0^L \frac{1}{2} (w_t^2 + c^2 w_x^2) dx = [c^2 w_t w_x]_0^L$$

$$w(x, 0) = 0 \Rightarrow w_x(x, 0) = 0, \quad 0 < x < L$$

$$w(0, t) = 0 \Rightarrow w_t(0, t) = 0$$

$$w(L, t) = 0 \Rightarrow w_t(L, t) = 0$$

$$\Rightarrow c^2 [w_t(L, t) w_x(L, t) - w_t(0, t) w_x(0, t)] = 0$$

$$\frac{d}{dt} \int_0^L \frac{1}{2} (w_t^2 + c^2 w_x^2) dx = 0$$

$$\int_0^L (w_t^2 + c^2 w_x^2) dx = \int_0^L [w_t^2(x, 0) + c^2 w_x^2(x, 0)] dx$$

$$= 0$$

$$\Rightarrow w_t = 0 \text{ and } w_x = 0$$

$$\Rightarrow w(x,t) = \text{constant}$$

$$w(0,t) = w(L,t) = 0$$

$$\Rightarrow \text{const} = 0$$

$$\Rightarrow w(x,t) = 0$$

$$\text{Thus, } u_1 = u_2$$

Q2: a) We solve

$$u_{xx} + \frac{\pi^2}{4} u = b$$

$$\Rightarrow u(x) = c_1 \cos \frac{\pi}{2} x + c_2 \sin \frac{\pi}{2} x + \frac{4b}{\pi^2}$$

$$u(0) = u(1) = 0$$

$$\Rightarrow c_1 = -\frac{4b}{\pi^2}$$

$$\frac{-4b}{\pi^2} \underbrace{\cos \frac{\pi}{2}}_0 + c_2 \underbrace{\sin \frac{\pi}{2}}_1 = -\frac{4b}{\pi^2}$$

$$\Rightarrow c_2 = -\frac{4b}{\pi^2}$$

$$\Rightarrow u(x) = -\frac{4b}{\pi^2} \left(\cos \frac{\pi}{2} x + \sin \frac{\pi}{2} x - 1 \right)$$

b) let $v = u - u_R$

$$\Rightarrow v_t = v_{xx} + \frac{\pi^2}{4} v$$

$$v(0,t) = v(1,t) = 0$$

$$v(x,0) = u_0 + \frac{4b}{\pi^2} \left(\cos \frac{\pi}{2} x + \sin \frac{\pi}{2} x - 1 \right) = f(x)$$

c) let $w = e^{\alpha t} v(x,t)$

$$w_t = (\alpha v + v_t) e^{\alpha t} \quad (1)$$

$$w_x = e^{\alpha t} v_x$$

$$w_{xx} = e^{\alpha t} v_{xx}$$

$$\text{From (1)} \Rightarrow v_t = e^{-\alpha t} w_t - \alpha v$$

$$v_t = v_{xx} + \frac{\pi^2}{4} v \Rightarrow e^{-\alpha t} w_t - \alpha v = e^{-\alpha t} w_{xx} + \frac{\pi^2}{4} v$$

$$\Rightarrow w_t = w_{xx} + e^{\alpha t} \left(\alpha + \frac{\pi^2}{4} \right) v$$

$$\text{let } \alpha = -\frac{\pi^2}{4}$$

$$\Rightarrow \begin{cases} w_t = w_{xx} \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = v(x,0) = f(x) \end{cases}$$

d) $w(x,t) = X(x)T(t)$

$$X T' = X'' T \Leftrightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 & T' + \lambda T = 0 \\ X(0) = X(1) = 0 \end{cases}$$

$$\lambda \leq 0 \text{ and } \lambda < 0 \Rightarrow X = 0$$

$$\lambda > 0, \lambda = \alpha^2 \Rightarrow X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$X(0) = 0 \Rightarrow c_1 = 0$$

$$X(1) = 0 \Rightarrow \sin \alpha = 0, \alpha = n\pi$$

$$\Rightarrow X_n(x) = c \sin n\pi x$$

$$T' + (n\pi)^2 T = 0$$

$$T_n(t) = c e^{-n^2 \pi^2 t}$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} B_n \sin n\pi x e^{-n^2 \pi^2 t}$$