

MATH 470.1 (Term 122)

Homework Exercises 4 (Sects. 6.1-6.16) Due date: May 15, 2013

1. Let  $S$  be the set of points  $(x, y)$  in  $\mathbb{R}^2$  with  $-2 \leq x < 3/2$ ,  $0 \leq y \leq 4$ . Determine all interior points of  $S$ , all boundary points of  $S$ , the closure of  $S$ , and whether  $S$  is open, closed, or neither open or closed. Is  $S$  connected? Is  $S$  bounded? Is  $S$  a domain?

2. Consider the following mixed boundary value problem for the Laplace equation in  $\mathbb{R}^2$

$$\Delta u(x, y) = 0, \quad 0 < x < a, \quad 0 < y < b, \quad (1)$$

with boundary conditions

$$u_y(x, 0) = 0, \quad u(x, b) = g(x), \quad 0 \leq x \leq a, \quad u(0, y) = f(y), \quad u(a, y) = 0, \quad 0 \leq y \leq b,$$

where  $f$  and  $g$  are given suitably smooth functions.

a.) Assume that  $p$  is a solution of problem (1) for  $g = 0$ , and that  $q$  satisfies the problem (1) for  $f = 0$ . Verify that the sum  $v = p + q$  is a solution of the full problem (1).

b.) Apply the method of separation of variables to obtain a solution of (1) for the specific boundary data.

$$f(y) = 5 \cos\left(\frac{11\pi}{2b}y\right) \quad \text{and} \quad g = 0. \quad (2)$$

Check explicitly that your solution satisfies all conditions of problem (1).

3. Prove the representation theorem for functions in  $\mathbb{R}^2$  by following the procedure that was detailed in the lecture for functions in  $\mathbb{R}^3$ :

a.) Derive Green's second identity

$$\iint_{\Omega} (u\Delta v - v\Delta u) dA = \oint_{\partial\Omega} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds, \quad (3)$$

which holds for functions  $u, v \in C^2(\Omega)$ , where  $\Omega \subset \mathbb{R}^2$  is a bounded domain with a piecewise smooth, closed boundary  $\partial\Omega$ .

b.) We know that the function

$$v(y) = \ln \frac{1}{|y - x|} \quad (4)$$

is harmonic in  $\Omega$  with the exception of point  $x$ . Apply the identity (3) to a domain formed by removing a ball around  $x$  from  $\Omega$ , with the function (4) inserted for  $v$  (note that all integrations are meant with respect to  $y$ ). Then apply manipulations, limits, and simplifications analogous to the  $\mathbb{R}^3$ -case, until you arrive at the representation theorem for  $\mathbb{R}^2$ :

$$u(x) = \frac{1}{2\pi} \oint_{\partial\Omega} \left\{ \frac{\partial u}{\partial n}(y) \ln \frac{1}{|y - x|} - u(y) \frac{\partial}{\partial n} \ln \frac{1}{|y - x|} \right\} ds - \frac{1}{2\pi} \iint_{\Omega} \Delta u(y) \ln \frac{1}{|y - x|} dA. \quad (5)$$

4. We know that the Laplace equation for the upper half-plane with boundary conditions on the  $x$ -axis

$$u_{xx} + u_{yy} = 0, \quad u(x, 0) = f(x), \quad x \in \mathbb{R}, \quad y > 0, \quad (6)$$

has the solution

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yf(\xi)}{y^2 + (\xi - x)^2} d\xi. \quad (7)$$

a.) Consider the equation in the upper left quadrant with homogeneous boundary conditions on the positive  $y$ -axis:

$$v_{xx} + v_{yy} = 0, \quad v(x, 0) = g(x), \quad v(0, y) = 0, \quad x < 0, \quad y > 0. \quad (8)$$

Use the result (6) to find an integral expression for the dependence of the solution of (8) on the function  $g$ , given on the negative  $x$ -axis.

b.) Now write a solution for the problem on the lower left quadrant:

$$w_{xx} + w_{yy} = 0, \quad w(x, 0) = h(x), \quad w(0, y) = 0, \quad x < 0, \quad y < 0. \quad (9)$$

Q.1:

$$S = \{(x, y) \in \mathbb{R}^2, -2 \leq x < 3/2, 0 \leq y \leq 4\}$$

$$\text{Interior of } S = \{(x, y) \in \mathbb{R}^2, -2 < x < 3/2, 0 < y < 4\}$$

$$\text{Boundary of } S = \{(x, y), (-2, y), (3/2, y), 0 \leq y \leq 4, (x, 0), (x, 4), -2 \leq x \leq 3/2\}$$

$$\text{Closure of } S = \{(x, y) \in \mathbb{R}^2, -2 \leq x \leq 3/2, 0 \leq y \leq 4\}$$

$S$  is not open, because points of the segment  $x = -2, 0 \leq y \leq 4$  are not interior points

$S$  is not closed, because it does not contain the segment  $x = 3/2, 0 \leq y \leq 4$  of its boundary.

$S$  is connected

$S$  is bounded,  $S \subset [-2, 3/2] \times [0, 4]$

$S$  is not a domain, because not open.

Q.2

$$\begin{cases} \nabla^2 u(x, y) = 0, & 0 < x < a, 0 < y < b \\ u_y(x, 0) = 0, & u(x, b) = g(x) \\ u(0, y) = f(y), & u(a, y) = 0 \end{cases}$$

a)

$$\begin{cases} \nabla^2 p(x, y) = 0 \\ p_y(x, 0) = 0, & p(x, b) = 0 \\ p(0, y) = f(y), & p(a, y) = 0 \end{cases}$$

$$\begin{cases} \nabla^2 q(x, y) = 0 \\ q_y(x, 0) = 0, & q(x, b) = g(x) \\ q(0, y) = 0, & q(a, y) = 0 \end{cases}$$

of course

$$\nabla^2(p+q) = \nabla^2 p + \nabla^2 q = 0$$

$$(p+q)_y(x,0) = p_y(x,0) + q_y(x,0) = 0$$

$$(p+q)(x,b) = p(x,b) + q(x,b) = g(x)$$

$$(p+q)(0,y) = p(0,y) + q(0,y) = f(x)$$

$$(p+q)(a,y) = p(a,y) + q(a,y) = 0$$

Thus,  $p+q$  is a solution to (1)

b)  $p = xY$

$$x''Y + xY'' = 0$$

$$\frac{x''}{x} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} x'' + \lambda x = 0 \\ x(a) = 0 \end{cases} \quad \begin{cases} Y'' - \lambda Y = 0 \\ Y'(0) = 0, Y(b) = 0 \end{cases}$$

•  $\lambda = 0$ ,  $Y = C_1 y + C_2$   
 $Y' = C_1 = 0$ ,  $Y(b) = C_2 = 0$   
 $\Rightarrow Y = 0$

•  $\lambda = \alpha^2$ ,  $Y'' - \alpha^2 Y = 0$   
 $Y = C_1 \cosh \alpha y + C_2 \sinh \alpha y$   
 $Y' = C_1 \alpha \sinh \alpha y + C_2 \alpha \cosh \alpha y$   
 $Y'(0) = 0 \Rightarrow C_2 = 0$   
 $Y(b) = 0 \Rightarrow C_1 = 0 \Rightarrow Y = 0$

•  $\lambda = -\alpha^2$ ,  $Y'' + \alpha^2 Y = 0$   
 $Y = C_1 \cos \alpha y + C_2 \sin \alpha y$   
 $Y'(y) = -C_1 \alpha \sin \alpha y + C_2 \alpha \cos \alpha y$   
 $Y'(0) = 0 \Rightarrow C_2 = 0$   
 $Y(b) = 0 \Rightarrow \cos \alpha b = 0$   
 $\alpha b = \frac{\pi}{2} + n\pi, n = 0, 1, 2, \dots$   
 $\alpha_n = \left(\frac{\pi}{2} + n\pi\right) \frac{1}{b}$

$$Y_n = C \cos\left(\frac{\pi}{2} + n\pi\right) \frac{y}{b}$$

$$X'' - \alpha_n^2 X = 0$$

$$X = C_1 e^{\alpha_n x} + C_2 e^{-\alpha_n x}$$

$$X(a) = 0 \Rightarrow C_1 e^{\alpha_n a} + C_2 e^{-\alpha_n a} = 0$$

$$C_1 = -C_2 e^{-2\alpha_n a}$$

$$\Rightarrow X(x) = -C_2 e^{-2\alpha_n a} e^{\alpha_n x} + C_2 e^{-\alpha_n x}$$

$$= -C_2 e^{-\alpha_n a} \left( e^{\alpha_n(x-a)} - e^{-\alpha_n(x-a)} \right)$$

$$= C_n \sinh \alpha_n(x-a)$$

$$\Rightarrow p(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi}{2} + n\pi\right) \frac{y}{b} \sinh\left(\frac{\pi}{2} + n\pi\right) \frac{(x-a)}{b}$$

$$p(0,y) = f(y) \Rightarrow$$

$$f(y) = \sum_{n=0}^{\infty} A_n \sinh\left(\frac{\pi}{2} + n\pi\right) \frac{a}{b} \cos\left(\frac{\pi}{2} + n\pi\right) \frac{y}{b}$$

$$\Rightarrow -A_n \sinh\left(\frac{\pi}{2} + n\pi\right) \frac{a}{b} = \frac{2}{b} \int_0^b f(y) \cos\left(\frac{\pi}{2} + n\pi\right) \frac{y}{b} dy$$

$$A_n = -\frac{2}{b \sinh\left(\frac{\pi}{2} + n\pi\right) \frac{a}{b}} \int_0^b f(y) \cos\left(\frac{\pi}{2} + n\pi\right) \frac{y}{b} dy$$

When  $f(y) = 5 \cos\left(\frac{11}{2} \frac{\pi}{b} y\right)$

$$\int_0^b \cos\left(\frac{11}{2} \frac{\pi}{b} y\right) \cos\left(\frac{\pi}{2} + n\pi\right) \frac{y}{b} dy =$$

$$\frac{1}{2} \int_0^b \left[ \cos \frac{\pi}{5} (n+6) \frac{y}{b} + \cos \frac{\pi}{5} (n-5) \frac{y}{b} \right] dy$$

$$= \frac{1}{2} \left[ \frac{\sin \frac{\pi}{5} (n+6) \frac{y}{b}}{\frac{\pi}{5} (n+6)} + \frac{\sin \frac{\pi}{5} (n-5) \frac{y}{b}}{\frac{\pi}{5} (n-5)} \right]_0^b$$

$$= \begin{cases} 0 & \text{if } n \neq 5 \\ \frac{b}{2} & \text{if } n = 5 \end{cases}$$

(3)