King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH 536 [Functional Analysis II]
Semester (122)

Exam II: April 22, 2013 Time allowed: 2hrs:

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(Q1) (a)Let P be an orthogonal projection on an inner product space X. Use an appropriate result to show that for each $z \in X, \| z \|^2 = \| x \|^2 + \| y \|^2$ where $x \in R(P)$ and $y \in N(P)$. Also check whether or not $R(P) = [N(P)]^{\perp}$.

(b)Let H be a Hilbert space and $P : H \to H$ be linear and bounded. The operator P is called projection on H if there is a closed subspace Y of H such that Y is the range of P and $Y^{\perp}$ is the null space of P and $P|Y = I$ (identity operator on Y.) If P is self-adjoint and idempotent, then prove that P is a projection on H.

(Q2) (a) Let P_1 and P_2 be projection on a Hilbert space H. Show that the sum $P = P_1 + P_2$ is a projection on H if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

(b) Let K be a closed and convex set in a Hilbert space $(H, \langle \cdot \rangle)$. If $z \in K$ is the projection of any $x \in H$, then show that $\langle x - z, y - z \rangle \leq 0$ for all $y \in K$.

(Q3) (a) Let X and Y be normed spaces. Show that a compact linear operator $T : X \to Y$ is continuous. If X is infinite dimensional , then check whether or not the identity operator on X is compact.

(b) Let T be a compact linear operator from a normed space X into a normed space Y. If a sequence $\{x_n\}$ in X weakly converges to x, then show that $\{Tx_n\}$ converges strongly in Y and has the limit $y = Tx$.

(Q4) (a) Let X and Y be normed spaces and $T : X \to Y$ be linear and compact. Then prove that $R(T)$, the range of T, is separable.

(b)Let X be a Banach space and $T \in B(X)$. Define $\rho(T)$, resolvent set of T. If $\lambda \in \rho(T)$, then show that $(T - \lambda I)^{-1} \in B(X)$.

(c) Find eigenvalues and eigen vectors of the projection operator $P : H \to Y$ where Y is a closed subspace of a Hilbert space H.