King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH 536 [Functional Analysis II]
Second Semester 2012-2013 (122)

Final Exam: May 18, 2013 Time allowed: 3hrs:

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(Q1) (a)Consider the sequence

$$\phi_1(x) = \frac{1}{\sqrt{2\pi}},\ \phi_{2n}(x) = \frac{1}{\sqrt{\pi}}\sin nx,\ \phi_{2n+1}(x) = \frac{1}{\sqrt{\pi}}\cos nx$$

in the usual space $L_2[-\pi,\pi]$.

Show that $\{\phi_n(x)\}$ is a complete orthonormal sequence. Apply parseval formula $\left(\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, \phi_n\rangle|^2\right)$ to the function $f(x) = x(-\pi \leq x \leq \pi)$ and show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(b) Use Parseval formula to prove that every separable infinite dimensional Hilbert space H is linearly isometric to $(l_2, \|\cdot\|_2)$.

(Q2) (a) Let H be a Hilbert space and $T \in L(H)$, the space of bounded linear operators from H into H and T^* be the adjoint operator of T. If T is regular, then show that T^* is regular and $(T^*)^{-1} = (T^{-1})^*$.

(b) If S and T are positive operators on a Hilbert space with $ST = TS$, then prove that ST is positive.

(Q3) (a) Let X be a normed space with dual X^*. Prove that the closed ball $B_1^* = \{f \in X^* : \| f \| \leq 1\}$ is compact with respect to weak*-topology.

(b) Let p_1 and p_2 be projections on a Hilbert space H. If $p = p_1p_2$ is a projection on H, then show that $p_1p_2 = p_2p_1$.

(Q4) (a) Let X and Y be normed spaces. If $T \in L(X,Y)$, then show that $T^* \in L(Y^*, X^*)$ and $\| T^* \| = \| T \|$.

(b) If X is a Banach space and $T \in L(X)$, then show that spectrum of $T, \sigma(T)$, is a nonempty set.

(Q5) (a) Let X be a Banach space and $T \in L(X)$. Then prove that spectrum $\sigma(p(T))$ of the operator

$$p(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \ldots + \alpha_0 I$$

is given by $p(\sigma(T) = \{\mu \in C : \mu = p(\lambda), \lambda \in \sigma(T)\}$.
i.e. Show that $\sigma(p(T) = p(\sigma(T))$.

(b) Let T be a bounded linear self-adjoint operator on a Hilbert space H and $p(x)$ be a polynomial of degree n in H. Use part(a), to show that

$$\| p(T) \| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|.$$

(Q6) (a) Use spectral mapping theorem for polynomials to show that spectrum set of an idempotent operator $T(\neq 0, I)$ on a Banach space is $\{0, 1\}$.

(b) Find spectrum set of the unitary operator u on a Hilbert space.

(c) Use Q2(a) to prove that for each $T \in L(H)$, we have $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.