1. **[20pts]** Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $U : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformations defined by $T(x, y) = (x - y, x, 2x + y)$ and $U(x, y, z) = (z - 2y, x - 2y + z)$.

(a) Prove that $UT$ is an isomorphism of $\mathbb{R}^2$ onto itself. Is $TU$ one-one? Is it onto? Justify your answer.

(b) Let $B$ be the standard basis of $\mathbb{R}^2$ and $B' = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Find $[T]_{B}^{B'}$.

(c) Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for a vector space $V$ and let $g : V \rightarrow W$ be a linear transformation. Prove that $\{g(\alpha_1), \ldots, g(\alpha_n)\}$ is a linearly independent subset of $W$ if and only if $g$ is one-one.

2. **[12pts]** Let $V$ be a finite-dimensional vector space over a field $F$.

(a) Prove that if $S$ is a nonempty subset of $V$, then $S^0 = (\text{span}(S))^0$.

(b) Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $v = (1, 0, -1, 2)$ and $w = (2, 3, 1, 1)$. Find all linear functionals $f : (x_1, x_2, x_3, x_4) \mapsto c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ that are in $W^0$.

3. **[20pts]** Let $V, W$ be finite-dimensional vector spaces over a field $F$.

(a) Prove that if $T \in \text{L}(V, W)$ then $\ker(T^t) = (\text{Im} T)^0$. Deduce that $T$ is onto if and only if $T^t$ is one-one.

(b) Prove that if $T^t$ is onto then $T$ is one-one.

(c) Prove that if $T$ is an isomorphism then $T^t$ is also an isomorphism.

4. **[20pts]** (a) Is the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$ (over $\mathbb{R}$) diagonalizable? Justify your answer.

(b) Find all values of $a$ for which the real matrix $\begin{bmatrix} 4 & 0 & 1 \\ 2 & a & 2 \\ 1 & 0 & 4 \end{bmatrix}$ is diagonalizable.

(c) Show that if an $n \times n$ matrix $C$ over a field $F$ is diagonalizable then its transpose $C^t$ is also diagonalizable.