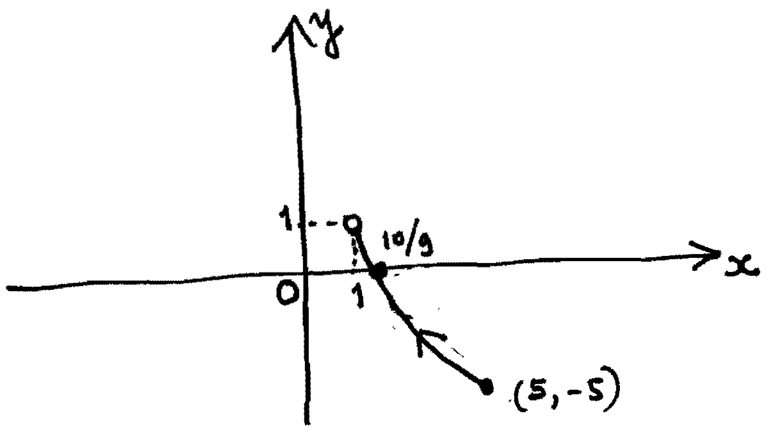
1. (a) Sketch the curve defined by the parametric equations $x = 1 + \frac{1}{t^2}$, $y = 1 - \frac{3}{t}$, $\left(\frac{1}{2} \le t < \infty\right)$
Clearly indicate:

- the coordinates of the initial point,
- any x- or y-intercepts,
- an arrow to show the direction of motion.

This can be done by plotting points and sketching a smooth curve containing these points, or by eliminating the parameter.
Eliminating t gives: $x = 1 + \left(\frac{y-1}{3}\right)^2$, so the curve is part of a parabola that starts at $(5,-5)$. There are no y-intercepts, and at $t=3$ we get an x-intercept $(x = \frac{10}{9})$. As $t \to \infty$, $x \to 1$ and $y \to 1$.



(b) Find $\frac{d^2y}{dx^2}$ at the point where $t = 1$ on the curve

$$x = 1 + \frac{1}{t^2}, \qquad y = 1 - \frac{3}{t}, \qquad \left(\frac{1}{2} \le t < \infty\right)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3/t^2}{-2/t^3} = -\frac{3t}{2},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(dy/dx) / (dx/dt) = \frac{-3/2}{-2/t^3} = \frac{3t}{4}$$

So, at $t = 1$, $\frac{d^2y}{dx^2} = \frac{3}{4}$.

2. (a) Replace the polar equation $r^2 = 8r\sin\theta - 2r\cos\theta$ by an equivalent Cartesian equation, and identify its graph.

$$x^2 + y^2 = 8y - 2x$$

$$x^2 + 2x + y^2 - 8y = 0$$

$$(x+1)^2 + (y-4)^2 = 17$$

Circle centered at $(-1, 4)$ with radius $\sqrt{17}$

(b) Find the distance between the two points with polar coordinates $\left(2, \frac{\pi}{4}\right)$ and $\left(-1, \frac{\pi}{6}\right)$

The Cartesian coordinates of the points are $(\sqrt{2}, \sqrt{2}), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

The distance required is

$$\sqrt{\left(\sqrt{2} + \frac{\sqrt{3}}{2}\right)^2 + \left(\sqrt{2} + \frac{1}{2}\right)^2} = \sqrt{2 + \sqrt{6} + \frac{3}{4} + 2 + \sqrt{2} + \frac{1}{4}} = \sqrt{5 + \sqrt{6} + \sqrt{2}}$$

3. Find the length of the parametric curve

$$x = -t^3, \qquad y = t^2, \qquad -1 \le t \le 0$$

Length is $L = \int_{-1}^{0} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-1}^{0} \sqrt{9t^4 + 4t^2}\, dt$

$= \int_{-1}^{0} |t| \sqrt{9t^2 + 4}\, dt = -\int_{-1}^{0} t \sqrt{9t^2 + 4}\, dt$

Set $u = 9t^2 + 4$, $du = 18t\, dt$, with $u = 13$ when $t = -1$ and $u = 4$ when $t = 0$. So

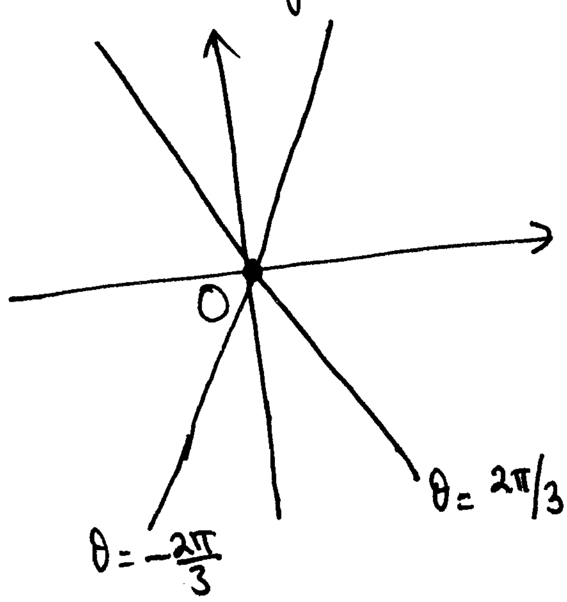
$$L = -\int_{13}^{4} \frac{1}{18} \sqrt{u}\, du = \frac{1}{18} \left[\frac{u\sqrt{u}}{3/2} \right]_4^{13} = \frac{1}{27} \left(13\sqrt{13} - 8 \right)$$

4. Sketch the polar curves given by

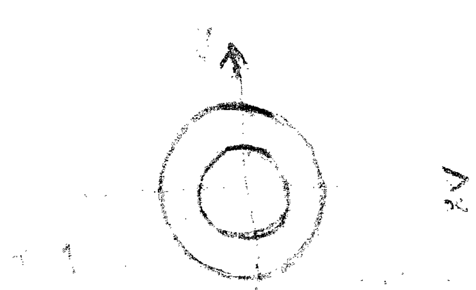
(a) $\theta^2 = \frac{4\pi^2}{9}$

(b) $r^2 - r - 2 = 0$

(a) $\theta = \pm \frac{2\pi}{3}$, two intersecting lines at the origin

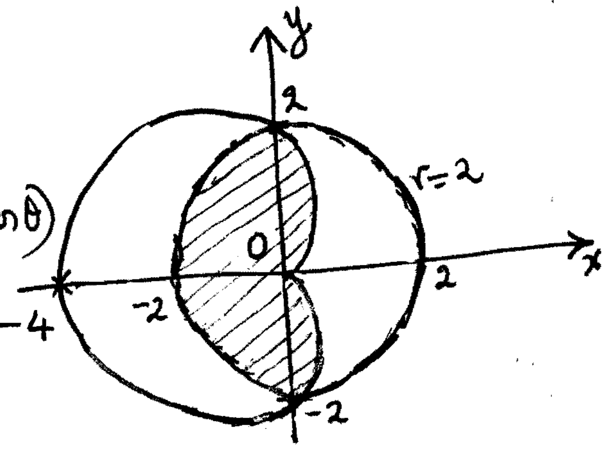


(b) $r^2 - r - 2 = 0$ is equivalent to $(r+1)(r-2) = 0$. So we have $r = -1$ or $r = 2$. The curves are the two circles with center $(0,0)$ & radii 1 and 2.



5. Find the area of the region shared by the curve $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

we want to find the area of the shaded region.



The points of intersection of the circle $r=2$ and the cardioid $r = 2(1-\cos\theta)$ occur at $\theta = \pm\frac{\pi}{2}$ $\left(2 = 2(1-\cos\theta) \Leftrightarrow \cos\theta = 0\right)$.

By symmetry, the area is

$$A = 2\int_0^{\pi/2} \frac{1}{2}\left(2(1-\cos\theta)\right)^2 d\theta + \frac{1}{2}\text{area of the circle}$$

$$= \int_0^{\pi/2} 4(1-2\cos\theta+\cos^2\theta)\,d\theta + 2\pi$$

$$= \int_0^{\pi/2} \left(4-8\cos\theta + 4\left(\frac{1+\cos 2\theta}{2}\right)\right) d\theta + 2\pi$$

$$= \int_0^{\pi/2} (6 - 8\cos\theta + 2\cos 2\theta)\,d\theta + 2\pi$$

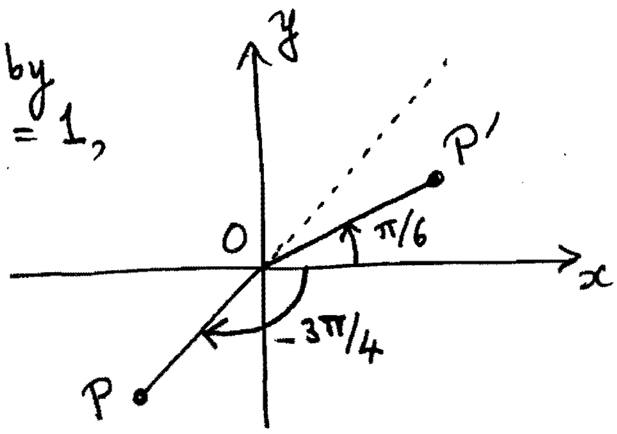
$$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + 2\pi$$

$$= 5\pi - 8$$

6. (a) Find the unit vector obtained by rotating the vector $\overrightarrow{OP}$ 165° counterclockwise about the origin, where P is the point $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

Let $\overrightarrow{OP'}$ be the vector obtained by rotating $\overrightarrow{OP}$. We have $\|\overrightarrow{OP'}\| = \|\overrightarrow{OP}\| = 1$, so $\overrightarrow{OP'}$ is a unit vector.

$\overrightarrow{OP'}$ makes an angle of $30°$ with the positive x-axis. So



$$\overrightarrow{OP'} = \left\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

(b) Let $\mathbf{u} = \langle -1, 1 \rangle$, $\mathbf{v} = \langle 2, -3 \rangle$, $\mathbf{w} = \langle 7, -6 \rangle$. Find the real numbers α and β such that $\frac{\alpha\mathbf{u} - \beta\mathbf{v}}{\|\mathbf{u} - \mathbf{v}\|} = \mathbf{w}$

$$u - v = \langle -3, 4 \rangle, \quad \|u - v\| = \sqrt{9 + 16} = 5$$

$$\alpha u - \beta v = \langle -\alpha - 2\beta, \alpha + 3\beta \rangle$$

Thus $\frac{\alpha u - \beta v}{\|u - v\|} = w$ implies $\begin{cases} -\alpha - 2\beta = 35 \\ \alpha + 3\beta = -30 \end{cases}$

Adding these two equations give $\beta = 5$, and then $\alpha = -30 - 15 = -45$

7. Find an equation of the line that passes through the point $P(1,-2)$ and is perpendicular to $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{u} = \langle 5, -9 \rangle$ and $\mathbf{v} = \langle -2, 2 \rangle$

Since $\text{proj}_v u$ is parallel to v, the line is perpendicular to v. So the line has an equation of the form

$$-2x + 2y = C,$$ where C is a constant.

Since $P(1,-2)$ is on the line, we have $-2-4 = C$ i.e. $C = -6$. An equation of the line is therefore

$$-2x+2y=-6, \text{ i.e. } y = x-3$$

8. Let $\mathbf{v} = \mathbf{j} - \mathbf{k}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{k}$. Find a point $P(x, y, z)$ in the first octant satisfying all of the following conditions:
$\left\|\overrightarrow{OP}\right\| = \sqrt{2}$, $\overrightarrow{OP}$ and $\mathbf{v} + \mathbf{w}$ are perpendicular, and the volume of the parallelepiped determined by $\overrightarrow{OP}$, $\mathbf{v}$ and $\mathbf{w}$ is equal to 2

(i) $\|\overrightarrow{OP}\|^2 = x^2 + y^2 + z^2 = 2$

(ii) $\overrightarrow{OP} \perp (v+w)$ iff $\overrightarrow{OP} \cdot (v+w) = 0$ i.e. $\langle x, y, z\rangle \cdot \langle -1, 1, 0\rangle = 0$,

i.e. $-x + y = 0$

(iii) The volume of the parallelepiped is

$$\left|(\overrightarrow{OP} \times v) \cdot w\right| = \left|\begin{matrix} x & y & z \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{matrix}\right| = |2x + z| = 2x + z = 2$$

($|2x+z| = 2x+z$ because x and z are positive, as P is in the first octant).

We then have the following system from (i), (ii) and (iii):

$$\left.\begin{matrix} x = y \\ 2x + z = 2 \\ x^2 + y^2 + z^2 = 2 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x = y \\ z = 2 - 2x \\ 2x^2 + (2-2x)^2 = 2 \end{matrix}\right.$$

The last equation is $6x^2 - 8x + 2 = 0$, or $3x^2 - 4x + 1 = 0$

with roots 1 and $\frac{1}{3}$. If $x = 1$, we get $y = 1$, $z = 0$, and

if $x = \frac{1}{3}$, we get $y = \frac{1}{3}$, $z = \frac{4}{3}$. So the point required

is $(1, 1, 0)$ or $\left(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}\right)$. [Taking positive coordinates for the first octant, only the point $(1/3, 1/3, 4/3)$ should be chosen. However, some authors take non-negative coordinates, in which case both points are valid.]