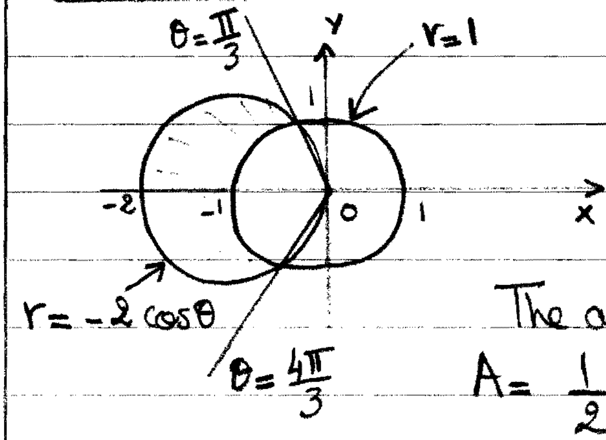
Math 201 Quiz 2 20/06/2013

1°) Find the area inside the circle $r=-2\cos\theta$ and outside the circle $r=1$

Solution:



The intersection points are given by: $-2\cos\theta=1 \Rightarrow \cos\theta=-\frac{1}{2}$ with $\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}$. So $\theta=\frac{2\pi}{3}$ or $\frac{4\pi}{3}$

The area of the desired region is:

$$A=\frac{1}{2}\int_{\frac{\pi}{3}}^{4\pi/3}\left[(-2\cos\theta)^2-1\right]d\theta$$

$$=\frac{1}{2}\int_{2\pi/3}^{4\pi/3}\left(4\cos^2\theta-1\right)d\theta=\frac{1}{2}\int_{2\pi/3}^{4\pi/3}\left[4\left(\frac{1+\cos 2\theta}{2}\right)-1\right]d\theta$$

$$=\frac{1}{2}\int_{2\pi/3}^{4\pi/3}\left(1+2\cos 2\theta\right)d\theta=\frac{1}{2}\left[\theta+\sin(2\theta)\right]_{2\pi/3}^{4\pi/3}$$

$$=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

2°) Find the center and the radius of the sphere:

$$2x^2+2y^2+2z^2+x+y+z=9$$

Solution:

$$2x^2+2y^2+2z^2+x+y+z=2\left[x^2+y^2+z^2+\frac{x}{2}+\frac{y}{2}+\frac{z}{2}\right]$$

$$=2\left[\left(x+\frac{1}{4}\right)^2-\frac{1}{16}+\left(y+\frac{1}{4}\right)^2-\frac{1}{16}+\left(z+\frac{1}{4}\right)^2-\frac{1}{16}\right]=9$$

Thus:

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{9}{2}+\frac{3}{16}=\frac{75}{16}$$

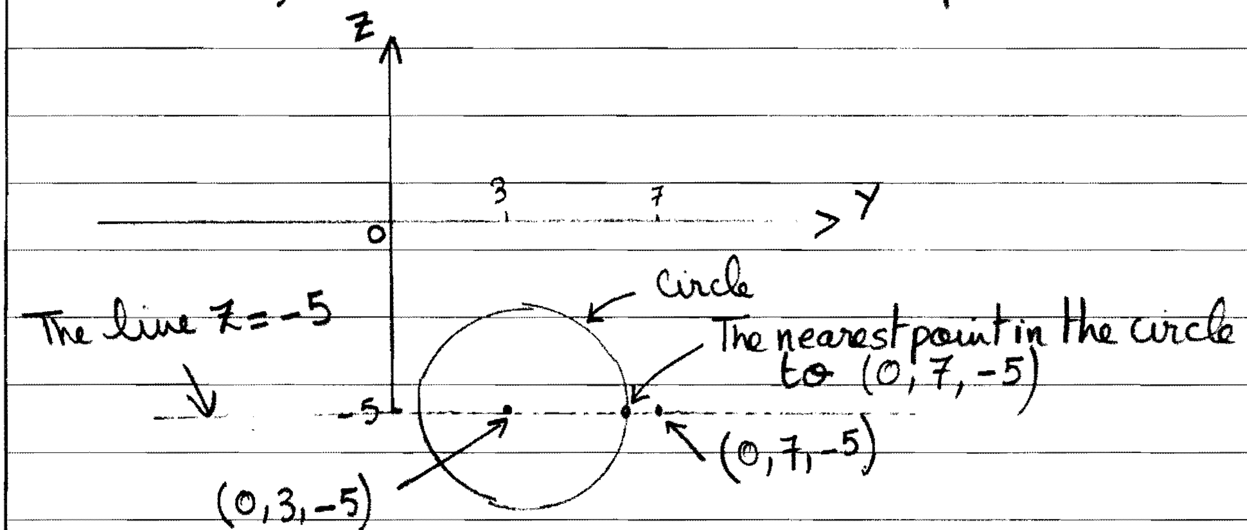
Hence the center is $\left(\frac{-1}{4},\frac{-1}{4},\frac{-1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$

3°) Find the point on the sphere $x^2+(y-3)^2+(z+5)^2=4$ that is nearest to the point $(0,7,-5)$.

Solution: The point $(0,7,-5)$ is in the yz-plane. The intersection of the sphere with the yz-plane is the circle: $(y-3)^2+(z+5)^2=4$, $x=0$

So the center is $(0,3,-5)$ and radius 2.

"$(0,3,-5)$ is also the center of the sphere".



So the z-coordinate of the nearest point is -5.

To find it y-coordinate, we use the equation of the circle (its x-coordinate is 0):

$$(y-3)^2+(-5+5)^2=4 \quad \text{i.e.} \quad y-3=\pm 2$$

and we take $y=3+2=5$. Therefore the nearest point to $(0,7,-5)$ is the point $(0,5,-5)$.

4°) Find $\text{proj}_{v}u$, where $u=i+j+k$ and $v=5j-3k$.

Solution:

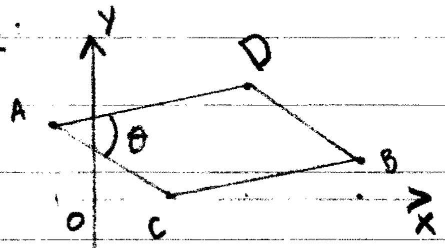
$$\text{proj}_{v}u=\left(\frac{u.v}{\|v\|}\right)\frac{v}{\|v\|}$$

$u.v=5-3=2$, $\|v\|^2=5^2+9=34$. Hence

$$\text{proj}_{v}u=\frac{2}{34}v=\frac{1}{17}(5j-3k)$$

5°) Find the area of the parallelogram with vertices $A(-1,2)$, $B(7,1)$, $C(2,0)$ and $D(4,3)$

Solution:



The Area of the parallelogram is given by:

Area $= \|\overrightarrow{AC}\| \, \|\overrightarrow{AD}\| \sin\theta$,

where θ is the angle between $\overrightarrow{AC}$ and $\overrightarrow{AD}$. θ is such that:

$\cos\theta = \dfrac{\overrightarrow{AC} \cdot \overrightarrow{AD}}{\|\overrightarrow{AC}\| \, \|\overrightarrow{AD}\|}$. Now: $\overrightarrow{AC} = \langle 3,-2 \rangle \Rightarrow \|\overrightarrow{AC}\| = \sqrt{13}$

$\overrightarrow{AD} = \langle 5,1 \rangle \Rightarrow \|\overrightarrow{AD}\| = \sqrt{26}$. So $\cos\theta = \dfrac{13}{\sqrt{26}\sqrt{13}} = \dfrac{1}{\sqrt{2}}$,

and hence $\theta = \dfrac{\pi}{4}$.

Therefore:

$$\text{Area} = \sqrt{13}\,\sqrt{26}\,\sin\left(\frac{\pi}{4}\right) = \sqrt{13}\,\sqrt{26}\,\frac{1}{\sqrt{2}} = 13$$

2nd method:

$$\text{Area} = \|\overrightarrow{AC} \times \overrightarrow{AD}\| = \|(3i - 2j) \times (5i + j)\| = \|13k\| = 13.$$

We consider them as vectors in $\mathbb{R}^3$