Math 201 Quiz 5 11/7/2013

Ex1: The derivative of $f(x,y,z)$ at a point P is greatest in the direction of $v = i + j - k$, and $(D_v f)_P = 2\sqrt{2}$.

a) Find $(\nabla f)_P$?

b) Find $(D_{i+j} f)_P$?

Solution: a) The derivative of $f(x,y,z)$ at P is greatest in the direction of $(\nabla f)_P$:

angle between $(\nabla f)_P$ and v

$$(D_v f)_P = \|(\nabla f)_P\| \left\| \frac{v}{\|v\|} \right\| \underbrace{\cos\theta}_{1} = 2\sqrt{2}$$

Thus $\|(\nabla f)_P\| = 2\sqrt{2}$. Therefore:

$$(\nabla f)_P = 2\sqrt{2} \underbrace{\frac{v}{\|v\|}}_{\text{unit vector in the direction of } v} = \frac{2\sqrt{2}}{\sqrt{3}} (i + j - k)$$

b) $$(D_{i+j} f)_P = (\nabla f)_P \cdot \frac{i+j}{\|i+j\|}$$

$$= \frac{2\sqrt{2}}{\sqrt{3}} [i + j - k] \cdot \frac{1}{\sqrt{2}} [i + j] = \frac{2}{\sqrt{3}} (1+1) = \frac{4}{\sqrt{3}}$$

Ex2: Find an equation for the plane that is tangent to the surface $z = \sqrt{y-x}$ at $(1,2,1)$.

Solution: Let $f(x,y,z) = \sqrt{y-x} - z$.

$$\frac{\partial f}{\partial x} = \frac{-1}{2\sqrt{y-x}} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,2,1)} = \frac{-1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y-x}} \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,2,1)} = \frac{1}{2}$$

$$\frac{\partial f}{\partial z} = -1 \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{(1,2,1)} = -1$$

$$\Rightarrow (\nabla f)_{(1,2,1)} = \langle -\tfrac{1}{2}, -\tfrac{1}{2}, -1 \rangle$$

An equation of the tangent plane is: $-\frac{1}{2}(x-1) - \frac{1}{2}(y-2) - (z-1) = 0$

Ex3: Let $f(x,y) = \ln x + \ln y$

a) Find $L(x,y)$ of $f(x,y)$ at $P_0(1,1)$

b) Find an upper bound of the error in the approximation $f(x,y) \approx L(x,y)$ over the rectangular region

$R: |x-1| \leqslant 0.2 \,,\ |y-1| \leqslant 0.2$.

Solution: a) $\frac{\partial f}{\partial x} = \frac{1}{x} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 1.$

$$\frac{\partial f}{\partial y} = \frac{1}{y} \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 1 \quad \text{Thus:}$$

$$L(x,y) = f(1,1) + \left.\frac{\partial f}{\partial x}\right|_{(1,1)}(x-1) + \left.\frac{\partial f}{\partial y}\right|_{(1,1)}(y-1) = x+y-2$$

b) $\frac{\partial^2 f}{\partial x^2} = \frac{-1}{x^2}$, $\frac{\partial^2 f}{\partial y^2} = -\frac{1}{y^2}$ and $\frac{\partial^2 f}{\partial x \partial y} = 0$

Over R, we have: $0.8 \leqslant x \leqslant 1.2 \Rightarrow 0.64 \leqslant x^2 \leqslant 1.44$

$$\Rightarrow \frac{1}{1.44} \leqslant \frac{1}{x^2} \leqslant \frac{1}{0.64} \Rightarrow -\frac{1}{0.64} \leqslant \frac{-1}{x^2} \leqslant -\frac{1}{1.44}.$$

Thus $\left|\frac{\partial^2 f}{\partial x^2}\right| \leqslant \frac{1}{0.64}$ over R.

Using a similar argument, we show that $\left|\frac{\partial^2 f}{\partial y^2}\right| \leqslant \frac{1}{0.64}$.
Hence the error $E(x,y)$ in the approximation $f(x,y) \approx L(x,y)$ satisfies:

$$|E(x,y)| \leqslant \frac{1}{2} \cdot \frac{1}{0.64}\left[|x-1|+|y-1|\right]^2 \leqslant \frac{1}{1.28}(0.2+0.2)^2$$

$$= 0.125$$

Ex4: Find all the local extrema and saddle points of $f(x,y) = (x^2+y^2)e^{-y}$.

<u>Solution</u> : •) $f_x(x,y) = 2x e^{-y}$

$f_y(x,y) = 2y e^{-y} - (x^2+y^2) e^{-y}$

Setting $f_x(x,y) = 0 = f_y(x,y)$ implies:

$$\begin{cases} 2x e^{-y} = 0 \\ \& \\ 2y e^{-y} - (x^2+y^2) e^{-y} = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ \& \\ (2-y)y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ \& \\ y = 0 \text{ or } y = 2 \end{cases}$$

Thus f has two critical points : $(0,0)$ and $(0,2)$

•) $f_{xx}(x,y) = 2e^{-y}$, $f_{yy}(x,y) = 2e^{-y} - 2ye^{-y} - 2ye^{-y} + (x^2+y^2) e^{-y} = (2 - 4y + x^2+y^2) e^{-y}$,

$f_{xy}(x,y) = -2x e^{-y}$

•) The Hessian at $(0,0)$ is:

$\Delta(0,0) = f_{xx}(0,0) f_{yy}(0,0) - (f_{xy}(0,0))^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$. Hence $(0,0)$ is a local minimum

•) The Hessian at $(0,2)$ is:

$\Delta(0,2) = (2e^{-2}).(-2e^{-2}) = -4e^{-2} < 0$. Therefore $(0,2)$ is a saddle point.