Chapter 3 MQR Markov Chain review

$p_{ij} = P(X_{n+1} = j|X_n = i) = \sum_{i=1}^{m} p_{ij} = 1$

$\pi_n = (\pi_1, \pi_2, \ldots, \pi_m)$

$r_{ij}^{(n)} = P(X_{n+r} = j|X_n = i; \pi_i^{(n)}, \pi_j^{(n)} = r_{ij}^{(n)}$ if cannot reenter state $i$ once left $i$

$\lambda_{ij}(s) = \text{force of transition. } \lambda_i(s) = \sum_{j=1}^{m} \lambda_{ij}(s) + \sum_{j=i+1}^{m} \lambda_{ij}(s) = \text{force of transitioning out.}$

$\frac{d}{dr}(r_{ij}^{(t)}) = \sum_{k \neq j} (r_{ik}^{(t)} \lambda_{kj}(t + r) - r_{ij}^{(t)} \lambda_{jk}(t + r)) = \sum_{k \neq j} (r_{ik}^{(t)} \lambda_{kj}(t + r) - r_{ij}^{(t)} \lambda_{jk}(t + r))$

Chapter 12 MQR Multiple Life Functions

The Joint Life and Last Survivor Statuses

$T_{xy} = \min(T_x, T_y)$

$S_{xy}(t) = \rho_x + \rho_y - \rho_{xy}$

Fundamental Symmetric Relations (from $\min(a, b) + \max(a, b) = a + b$)

Two Independent Lifetimes

$\mu_{x+y+t} = \mu_x + \mu_y + t$

$\rho_x + \rho_y - \rho_{xy}$ or $S_x(t) + S_y(t) = S_{xy}(t) + S_{xy}(t)$ (Survival Function)

$\rho_{xy} + \rho_{xy} - \rho_{xy}$ or $S_{xy}(t) + S_{xy}(t) = F_x(t) + F_y(t)$ (Distribution Function)

$\mu_{x+y+t} = \mu_x + \mu_y + t$

$T_{x+y} + T_{xy} = T_x + T_y$

Force of Mortality of the Last Survivor Status: $\mu_{x+y+t} = \frac{\rho_x \mu_x + \rho_y \mu_y + t \rho_{xy} \mu_{x+y+t}}{\rho_{xy} \mu_{x+y+t}}$

Mean, Variance and Covariance of Two Lifetimes

$\overline{\epsilon}_{xy} = E(T_{xy}) = \int_0^\infty t \cdot f_{xy}(t) dt = \int_0^\infty t \rho_{xy} dt$

$\overline{\epsilon}_{xy} = E(T_{xy}) = \int_0^\infty t \cdot f_{xy}(t) dt = \int_0^\infty t \rho_{xy} dt$

$\overline{\epsilon}_{xy} = E(K_{xy}^*) = \sum_{k=1}^{\infty} k \rho_{xy}$

$E(T_x) + E(T_y) = E(T_{xy}) - E(T_x) \cdot E(T_y)$

$\text{Cov}(T_x, T_y) = E(T_x \cdot T_y) - E(T_x) \cdot E(T_y)$

$\text{Var}(T_x + T_y) = \text{Var}(T_x) + \text{Var}(T_y) - 2 \{E(T_x) - E(T_{xy})\} (E(T_y) - E(T_{xy}))$

$\text{Cov}(T_x, T_{xy}) = \text{Cov}(T_x, T_y) + [E(T_x) - E(T_{xy})] \cdot [E(T_y) - E(T_{xy})]$ if $T_x$ & $T_y$ independent

$E(T_{xy}) = \int_0^\infty t \rho_{xy} dt$

Symmetrical Relation between Joint and Last Survivor Continuous Insurance

$A_{xy} + A_{xy} = A_x + A_y$

Similar relations hold for $n$-year term, pure endowment, and endowment insurance.

Covariance between Joint and Last Survivor Benefits

$\text{Cov}(V_{T_x}, V_{T_y}) = \text{Cov}(V_{T_x}, V_{T_y}) + (A_x - A_{xy}) (A_y - A_{xy})$

Similarly hold for $n$-year term, pure endowment, and endowment insurance.

1. Relation between Insurances and Annuities

$\pi_{xy} = \frac{1 - A_{xy}}{d}, \quad \pi_{xy} = \frac{1 - A_{xy}}{d}$

2. Fully Discrete Insurances and Annuities
3. Reversionary Annuities (payment only when one life fails until the other also fails)

Payment to \( y \) when \( x \) has failed: \( a_{x|y} = \sum_{k=1}^{\infty} v^k (kq_x - k p_y) = \sum_{k=1}^{\infty} v^k (kp_y - k p_x) = a_y - a_{xy} \)

\( n \)-yrs (at most) pmt to \( x \) when \( y \) has failed: \( a_{y|x:n} = \sum_{k=1}^{n} v^k (kq_x - k p_y) = \sum_{k=1}^{n} v^k (kp_y - k p_x) = a_{y:n} - a_{xy:n} \)

Continuous Payment to \( T \) non-UDD (Woolhouse formula):

\[
P(a_{y|x}) = \frac{a_{y|y}}{a_{y|y} + a_{y|x} + a_{xy}}.
\]

\( tV(a_{y|x}) = \left\{ \begin{array}{ll} a_{y+t|x} + tP(a_{y|x}) & \text{both survives} \\ 0 & \text{if } (x) \text{ survives and } (y) \text{ fails} \\ a_{y|x} & \text{if } (x) \text{ fails and } (y) \text{ survives} \end{array} \right. \)

4. Contingent Insurance

\[
\begin{align*}
\overline{A}_{xy} &= \int_0^\infty v^t \cdot p_x \mu_{x+t} \, dt \\
\mathcal{A}_{xy} &= \int_0^\infty v^t \cdot t \mu_{x+t} \cdot (1 - t p_y) \, dt = \overline{A}_{xy} - \overline{A}_{xy}.
\end{align*}
\]

\[\overline{A}_x + \overline{A}_y = A_{xy}, \quad \overline{A}_x + \overline{A}_y = \overline{A}_{xy}.\]

5. \( m \)-thly payable multiple life benefits

\[
\alpha(m) = \frac{\overline{a}_{x|y} - \overline{a}_{xy}}{\overline{a}_{xy} - \overline{a}_{xy}} = \frac{s(m)}{d(m)} - 1 = \frac{i - i(m)}{i(m)}
\]

\[
\beta(m) = \frac{s(m) - 1}{d(m)} = \frac{i - i(m)}{i(m)} - 1.
\]

Non-UDD (Woolhouse formula):

\[
\overline{a}_{x|y} \approx \overline{a}_{xy} \approx \overline{a}_{xy} = \frac{2m - 1}{12m^2} \left( 8 + \delta + \mu_{xy} \right).
\]

Premium and Reserves

\[
P_{xy} = \frac{\overline{A}_{xy}}{\overline{a}_{xy}} \quad P_{xy} = \frac{\overline{A}_{xy}}{\overline{a}_{xy}} \quad iV_{xy} = A_{x+t:y+t} - P_{xy} \cdot \overline{a}_{x+t:y+t}
\]

\[
\overline{A}_{xy} = \left\{ \begin{array}{ll} A_{x+t:y+t} - \overline{A}_{xy} \cdot \overline{a}_{x+t:y+t} & \text{if } (x) \text{ and } (y) \text{ survive} \\ A_{x+t} - \overline{A}_{xy} \cdot \overline{a}_{x+t} & \text{if } (x) \text{ survives and } (y) \text{ fails} \\ A_{y+t} - \overline{A}_{xy} \cdot \overline{a}_{y+t} & \text{if } (x) \text{ fails and } (y) \text{ survives} \end{array} \right. \]

Dependent Life Models - Common Shock Model

\[
\mu_{x+t} = \mu_{x+t} + \lambda, \quad \mu_{y+t} = \mu_{y+t} + \lambda, \quad \mu_{y+t} = \mu_{y+t} + \lambda
\]

An Exponential Common Shock Model with Constant Force of Transitions (From ACTEX MLC manual)

\[
T_x \sim \exp(\mu_x + \lambda), \quad T_y \sim \exp(\mu_y + \lambda), \quad T_{xy} \sim \exp(\mu_x + \mu_y + \lambda)
\]

Chapter 13 MQR Multiple Decrement Models: Theory

OBJECTIVES: 1. To understand the concept of a multiple decrement table
2. To understand the force of decrement
3. To construct a multiple decrement model using associated single decrements and to apply various assumptions to calculate rates for discrete jumps.

13.1 Discrete Multiple Decrement Models

\[
q^{(r)}_x = q^{(1)}_x + q^{(2)}_x + \ldots + q^{(m)}_x = \sum_{j=1}^{m} q^{(j)}_x
\]

\[
p^{(r)}_x = 1 - q^{(r)}_x
\]

\[
n_p(x) = 1 - n_q(x)
\]

\[
e'_{x:n} = e^{(r)}_x \cdot n_q(x)
\]

\[
\ell_x^{(r)} = \sum_{j=1}^{m} \ell^{(j)}_x
\]

\[
d^{(r)}_x = \sum_{j=1}^{m} d^{(j)}_x = e^{(r)}_x \cdot q^{(r)}_x
\]

\[
n_d^{(j)}_x = \sum_{t=0}^{n-1} d^{(j)}_x = \ell^{(r)}_x \cdot n_q^{(j)}_x
\]
13.2 Theory of Competing Risks

\[ n_d^{(r)} = \sum_{j=1}^{m} n_d^{(j)} = \ell_x^{(r)} \cdot n_q^{(r)} \quad (13.5a \text{ & } 13.7d) \]
\[ nq_x^{(r)} = \sum_{j=1}^{n} nq_x^{(j)} \quad (13.5b) \]

13.1.2 Random Variable Analysis

The joint probability function of \( K_x^* \) and \( J_x \) is

\[ \text{Pr} (K_x^* = k \cap J_x = j) = k^{-1} q_x^{(j)} = \frac{d_{x+k-1}^{(j)}}{\ell_x^{(r)}} \quad (13.8) \]

The marginal probability functions are

i) \[ \text{Pr} (K_x^* = k) = \sum_{j=1}^{m} \text{Pr} (K_x^* = k \cap J_x = j) = k^{-1} q_x^{(j)} = \frac{d_{x+k-1}^{(j)}}{\ell_x^{(r)}} \]

ii) \[ \text{Pr} (J_x = j) = \sum_{k=1}^{\infty} \text{Pr} (K_x^* = k \cap J_x = j) = \sum_{k=1}^{\infty} \frac{d_{x+k-1}^{(j)}}{\ell_x^{(r)}} \]

13.2 Theory of Competing Risks

\[ nq_x^{(j)} \geq nq_x^{(j)} \quad (13.11) \]

13.3 Continuous Multiple Decrement Models

\[ \mu_{x+t}^{(j)} = \frac{d}{dt} p_x^{(j)} \quad (13.12a) \]
\[ \mu_x^{(r)} = \frac{d}{dt} p_x^{(r)} \quad (13.13a) \]
\[ \sigma_x^{(j)} = \exp \left( - \int_0^t \mu_{x+s}^{(j)} ds \right) \quad (13.12b) \]
\[ \sigma_x^{(r)} = \exp \left( - \int_0^t \mu_{x+s}^{(r)} ds \right) \quad (13.13b) \]
\[ f_x(t) = p_x^{(j)} \cdot \mu_{x+t}^{(j)} \quad (13.14) \]
\[ F_x(t) = \text{Pr} \left[ T_x^* \leq t \right] = \int_0^t f_x(s) ds = \int_0^t \sigma_x^{(j)} \cdot \mu_{x+s}^{(j)} ds \quad (13.15) \]
\[ \mu_{x+t}^{(r)} = \frac{d}{dt} p_x^{(r)} \quad (13.16) \]
\[ \sigma_x^{(j)} = \int_0^t f_{T_x^*} (s) ds = \int_0^t p_x^{(j)} \cdot \mu_{x+s}^{(j)} ds \quad (13.17) \]
\[ \mu_{x+t}^{(r)} = \frac{d}{dt} p_x^{(r)} \quad (13.18) \]
\[ \sigma_x^{(j)} = \int_0^t s_{x+s}^{(j)} ds = \int_0^t p_x^{(j)} \cdot \mu_{x+s}^{(j)} ds \quad (13.19) \]

13.4.1 Uniform Distribution of Decrement Rates in the Multiple Decrement Context

\[ \mu_x^{(j)} = \frac{t \cdot q_x^{(j)}}{1 - t \cdot q_x^{(r)}} \quad (13.20) \]
\[ \mu_x^{(r)} = \frac{t \cdot q_x^{(r)}}{1 - t \cdot q_x^{(r)}} \quad (13.21) \]
\[ \mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{1 - t \cdot q_x^{(r)}} \quad (13.22) \]
\[ \mu_{x+t}^{(r)} = \frac{q_x^{(r)}}{1 - t \cdot q_x^{(r)}} \quad (13.23) \]
\[ \sigma_x^{(j)} = \exp \left[ \frac{q_x^{(j)}}{q_x^{(r)}} \cdot \ln \left( 1 - t \cdot q_x^{(r)} \right) \right] = \frac{\mu_x^{(j)} \cdot q_x^{(j)}}{q_x^{(r)}} \quad (13.24) \]

13.4.2 Uniform Distribution in the Associated Single-Decrement Tables

\[ \mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{1 - t \cdot q_x^{(r)}} \quad (13.25) \]
\[ \mu_{x+t}^{(r)} = \frac{q_x^{(r)}}{1 - t \cdot q_x^{(r)}} \quad (13.26) \]

Double decrement case:
\[ q_x^{(1)} = q_x^{(2)} = \frac{1}{2} \cdot \frac{q_x^{(1)}}{q_x^{(r)}} \quad (13.27) \]

Triple decrement case:
\[ q_x^{(1)} = q_x^{(2)} = q_x^{(3)} = \frac{1}{3} \cdot \frac{q_x^{(1)}}{q_x^{(r)}} \quad (13.28) \]

Miscellaneous Results (From ACTEX MLC Manual)

1. Assumptions on the single decrement table.
   - Backing out the Unprimed Rates from Primed Rates
   \[ s_{x}^{(i)} = \int_0^t s_{x+s}^{(j)} ds = \int_0^t \prod_{j=1, j \neq i} s_{x+s}^{(j)} \cdot p_x^{(j)} \cdot \mu_{x+t}^{(j)} dt \]

2. Constant Force Assumption for Multiple Decrement
   \[ \text{For any } t \in [0, 1] \text{ and integer-valued } x, \]
(i) \( t_p x^{(r)} = \left[ p_x^{(r)} \right]^t \) (survival probability for fractional ages)

(ii) Ratio Property: \( \frac{t_i x^{(i)}_{x+t}}{t_i x^{(i)}} = \frac{\mu^{(i)}_{x+t}}{\mu^{(i)}_x} \) for any \( s \in [0, 1] \) (To get unprimed rates from (i))

(iii) Partition Property: \( t_p x^{(i)} = \left[ t_p x^{(r)} \right] q^{(i)}_r / q^{(r)}_r \) (To get primed rates from unprimed rates \( t_p x^{(i)} \) and \( t_p x^{(r)} \))

Discrete jumps: Handling Both Discrete and Continuous Decrement

1) \( s q^{(i)}_x = \int_0^x \left[ \prod_{j=1, j \neq i}^{m} t_{p x}^{(j)} \right] s q^{(i)}_x^{(i)} / p^{(i)}_x t^{(i)} \) holds when decrement \( i \) is continuous.

2) \( s q^{(i)}_x = \sum_{t_k \leq x} \left[ \prod_{j=1, j \neq i}^{m} t_{p x}^{(j)} \right] \Delta (t_k q^{(i)}_x) \) holds when decrement \( i \) is discrete.

Here \( t_k \) are the jump times and \( \Delta (t_k q^{(i)}_x) \) is the jump size at time \( t_k \).

14.1 Actuarial Present Value

\[ A_x = \sum_{k=1}^\infty v^k \cdot \Pr(K_x^* = k) \]  \hspace{1cm} (14.1)

\[ A^{(j)}_x = \sum_{k=1}^\infty v^k \cdot \Pr(K_x^* = k \land J_x = j) \]  \hspace{1cm} (14.2)

If the time and cause of decrement are independent,

\[ A^{(j)}_x = \sum_{k=1}^\infty v^k \cdot \Pr(K_x^* = k) \cdot \Pr(J_x = j) \]  \hspace{1cm} (14.3a)

or \( A^{(j)}_x = \sum_{k=1}^\infty v^k \cdot k-1 p^{(r)}_x \cdot q^{(j)}_{x+k-1} \)  \hspace{1cm} (14.3b)

For benefit paid at the instant of failure \( \overline{A}_x^{(j)} = \int_0^\infty v^t \cdot t p^{(r)}_x \cdot \mu^{(j)}_{x+t} \) \hspace{1cm} (14.4)

14.2 Asset Shares \( [0] AS + G(1 - r_1 - e_1] (1 + i) \) \hspace{1cm} (14.5a)

so \( [0] AS = \left[ b^{(1)}_1 \cdot q^{(1)}_x + b^{(2)}_2 \cdot q^{(2)}_x + 1 AS \cdot p^{(r)}_x \right] \) \hspace{1cm} (14.5b)

In general, \( [k-1] AS + G(1 - r_k - e_k] (1 + i) \) \hspace{1cm} (14.6a)

so \( k AS = \left[ b^{(1)}_k \cdot q^{(1)}_{x+k-1} + b^{(2)}_k \cdot q^{(2)}_{x+k-1} + k AS \cdot p^{(r)}_{x+k-1} \right] \) \hspace{1cm} (14.6b)

14.3 Non-Forfeiture Options

14.3.1 Cash Value \( A^{1}_{x+t,n} \)

14.4 Multi State Model Representation

14.4.2 The Total and Permanent Disability Model

\[ h \overline{A}^{(j)}_x = \int_0^\infty v^r \cdot t p^{(r)}_x \cdot \mu^{(j)}_{x+t} \) \hspace{1cm} (14.12a)

\[ d \overline{A}^{(j)}_x = \int_0^\infty v^s \cdot s p^{(0)}_1 \cdot \lambda^{(r)}_1 \) \hspace{1cm} (14.12b)

\[ h \overline{A}^{(d)}_x = \int_0^\infty v^r \cdot t p^{(r)}_x \cdot \mu^{(d)}_{x+t} \) \hspace{1cm} (14.13a)

\[ d \overline{A}^{(d)}_x = \int_0^\infty v^s \cdot s p^{(0)}_2 \cdot \lambda^{(r)}_2 \) \hspace{1cm} (14.13b)

\[ h \overline{A}^{(d)}_{x+t} = \int_0^\infty v^r \cdot t p^{(r)}_x \cdot \mu^{(d)}_{x+t} \) \hspace{1cm} (14.14a)

\[ d \overline{A}^{(d)}_{x+t} = \int_0^\infty v^s \cdot s p^{(0)}_1 \cdot \lambda^{(r)}_2 \) \hspace{1cm} (14.14b)

\[ h \overline{A}^{(d)}_{x+n} = \int_0^\infty v^r \cdot t p^{(r)}_x \cdot \mu^{(d)}_{x+n} \) \hspace{1cm} (14.15a)

\[ d \overline{A}^{(d)}_{x+n} = \int_0^\infty v^s \cdot s p^{(0)}_2 \) \hspace{1cm} (14.15b)
14.4.3 Disability Model Allowing For Recovery \( f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x \) (14.16)

\[
\frac{d}{dr} rP_{11}^{(f)} = rP_{12}^{(f)} \cdot \lambda_{12}(t + r) - rP_{11}^{(f)} \cdot \lambda_{12}(t + r) \quad \text{(at } k=2) \\
+ rP_{12}^{(f)} \cdot \lambda_{31}(t + r) - rP_{11}^{(f)} \cdot \lambda_{13}(t + r) \quad \text{(at } k=3) \\
= rP_{12}^{(f)} \cdot \lambda_{12}(t + r) - rP_{11}^{(f)} \cdot [\lambda_{12}(t + r) + \lambda_{13}(t + r)], \quad (14.17)
\]

\[
\frac{d}{dr} rP_{12}^{(f)} = rP_{11}^{(f)} \cdot \lambda_{12}(t + r) - rP_{12}^{(f)} \cdot \lambda_{21}(t + r) \quad \text{(at } k=1) \\
+ rP_{12}^{(f)} \cdot \lambda_{32}(t + r) - rP_{12}^{(f)} \cdot \lambda_{23}(t + r) \quad \text{(at } k=3) \\
= rP_{11}^{(f)} \cdot \lambda_{12}(t + r) - rP_{12}^{(f)} \cdot [\lambda_{21}(t + r) + \lambda_{23}(t + r)] \quad (14.18)
\]

\[
r_{\Delta}P_{11}^{(0)} \approx r_{p_{ij}}^{(0)} + \frac{d}{dr} rP_{ij}^{(f)} \cdot \Delta r \quad (14.19)
\]

\[
r_{\Delta}P_{12}^{(0)} \approx r_{p_{ij}}^{(0)} + \Delta r \left\{ r_{P_{12}} \cdot \lambda_{21}(r) - r_{P_{11}}^{(0)} \cdot [\lambda_{12}(r) + \lambda_{13}(r)] \right\}, \quad (14.20)
\]

\[
r_{\Delta}P_{12}^{(0)} \approx r_{p_{ij}}^{(0)} + \Delta r \left\{ r_{P_{11}}^{(0)} \cdot \lambda_{12}(r) - r_{P_{12}}^{(0)} \cdot [\lambda_{21}(r) + \lambda_{23}(r)] \right\}, \quad (14.21)
\]

14.5.4 Thiele’s Differential Equation in the Multiple Decrement Case

\[
\bar{a}_{x}^{(r)} = \int_{0}^{\infty} v^{r} \cdot r_{p_{x}}^{(r)} \cdot dr \quad (14.22) \quad \bar{P} = \frac{APV_{B}^{(0)}}{\bar{a}_{x}^{(r)}} = \frac{APV_{x}^{(f)}}{\bar{a}_{x}^{(r)}} = \frac{\sum_{j=1}^{m} APV_{V_{x}^{(j)}}}{\bar{a}_{x}^{(r)}} \quad (14.23)
\]

\[
\frac{d}{dt} \bar{a}_{x+t} = \int_{0}^{\infty} v^{*} \cdot s_{x+t}^{(f)} \cdot dt \quad (14.24) \quad \frac{d}{dt} \bar{a}_{x+t} = \bar{P} + \delta \cdot \bar{a}_{x+t} - \mu_{x+t}^{(f)} \cdot (\bar{a}_{x+t}^{(f)} - \bar{a}_{x+t}^{(r)}) \quad (14.25)
\]

\[
\frac{d}{dt} \bar{a}_{x+t} = \delta \cdot \bar{a}_{x+t} - \bar{P} + \delta \cdot \bar{a}_{x+t} - \mu_{x+t}^{(f)} \cdot (\bar{a}_{x+t}^{(f)} - \bar{a}_{x+t}^{(r)}) \quad (14.26)
\]

\[
\frac{d}{dt} \bar{a}_{x+t} = \delta \cdot \bar{a}_{x+t} - \bar{P} + \delta \cdot \bar{a}_{x+t} - \mu_{x+t}^{(f)} \cdot (\bar{a}_{x+t}^{(f)} - \bar{a}_{x+t}^{(r)}) \quad (14.27)
\]

\[
\frac{d}{dt} \bar{a}_{x+t} = \delta \cdot \bar{a}_{x+t} - \bar{P} + \delta \cdot \bar{a}_{x+t} - \mu_{x+t}^{(f)} \cdot (\bar{a}_{x+t}^{(f)} - \bar{a}_{x+t}^{(r)}) \quad (14.28)
\]

14.5.5 Defined Benefit (DB) Pension Plans

14.5.1 Normal Retirement (NR) Benefits

Projected Annual Benefit \( PAB_{x} = 0.01p \cdot YOS_{x} \cdot FAS_{x} \) (14.29)

Final Annual Salary \( FAS_{x} = \frac{1}{3} \left( S_{x-3} + S_{x-2} + S_{x-1} \right) \cdot CAS_{x} \) (14.30)

Projected Aggregate Salary \( PAS_{x} = \frac{1}{S_{x}} \sum_{k=x}^{x-1} S_{k} \cdot CAS_{x} \) (14.31)

Projected Annual Retirement Benefit: \( PAB_{x} = 0.01p \cdot PAS_{x} \) (14.32)

APV of the projected benefit, at age \( x \): \( APV_{x}^{NR} = PAB_{x} \cdot v^{x-r} \cdot z_{-x}^{(r)} \cdot r_{d_{x}^{(12)}} \) (14.33)

14.5.2 Early Retirement (ER) Benefits

\[
APV_{x}^{ER} = \sum_{y=60}^{64} PAB_{y+1/2} \cdot \left[ 1 - 0.05 \left( 65 - y - \frac{1}{2} \right) \right] \cdot v^{y+1/2-x} \cdot y_{35}^{(r)} \cdot P_{35}^{(r)} \cdot q_{y}^{(r)} \cdot r_{d_{y+1/2}^{(12)}} \quad (14.34)
\]

14.5.3 Withdrawal and other Benefits

Assuming a 5-year vesting rule and assuming employees take their withdrawal benefit at NRA, the APV at age 35 is

\[
APV_{x}^{W} = \sum_{y=35}^{50} PAB_{y+1/2} \cdot v^{30-y} \cdot y_{35}^{(r)} \cdot q_{y}^{(w)} \cdot 65-y-1/2 \cdot P_{y+1/2} \cdot r_{d_{y+1/2}^{(12)}} \quad (14.35)
\]

14.5.4 Funding and Reserving

Normal Cost (Early Age) \( NC_{x}^{EAN} = \frac{APV_{x}^{T}}{\bar{a}_{x}^{(r)} \cdot z_{-x}} \) (14.36)

\[
\bar{V}_{x}^{T} = APV_{x}^{T} - NC_{x}^{EAN} \cdot \bar{a}_{x+t}^{(r)} \quad (14.37a) \quad \text{or retrospectively as} \quad \bar{V}_{x}^{T} = NC_{x}^{EAN} \cdot r_{a_{x+t}}^{(12)} \quad (14.37b)
\]

APV of the benefit accrued between ages \( x \) and \( x+1 \): \( APV_{x}^{NR} = (AB_{x+1} - AB_{x}) \cdot v^{x-r} \cdot z_{-x}^{(r)} \cdot r_{d_{x}^{(12)}} \) (14.38)

14.5 Gain and Loss Analysis

Profit with all anticipated factors:

\[
P(0) = [v^{G} \cdot (1 - r_{t+1}) - e_{t+1}] \left( \frac{q^{(1)}_{t+1} + s^{(1)}_{t+1}}{q^{(1)}_{t+1} + s^{(1)}_{t+1}} \right) \cdot q^{(t)}_{t+1} \left( q^{(2)}_{t+1} + s^{(2)}_{t+1} \right) \cdot q^{(2)}_{t+1} + p^{(r)}_{t+1} \cdot t+1 \bar{V} \quad (14.39)
\]

Profit with some actual experience in place of anticipated factors:
\[ P(1) = (14.39) \] with all anticipated factors except actual value for 1 factor.
\[ P(2) = (14.39) \] with all anticipated factors except actual value for 2 factors
\[ P(3) = (14.39) \] with all anticipated factors except actual value for 3 factors.
\[ P(4) = (14.39) \] with all anticipated factors except actual value for 4 factors.

Gain from factor whose gain is calculated first is \( G_{F_1} = P(1) - P(0) \) \hspace{1cm} (14.40a)
Gain from factor whose gain is calculated second is \( G_{F_2} = P(2) - P(1) \) \hspace{1cm} (14.40b)
Gain from factor whose gain is calculated third is \( G_{F_3} = P(3) - P(2) \) \hspace{1cm} (14.40c)
Gain from factor whose gain is calculated fourth is \( G_{F_4} = P(4) - P(3) \) \hspace{1cm} (14.40d)

Total gain \( G = G_{F_1} + G_{F_2} + G_{F_3} + G_{F_4} = P(4) - P(0) \) \hspace{1cm} (14.41)

When death occurs throughout year but withdrawal only at end of year, the anticipated profit expression is
\[ P(0) = [V + G(1 - r_{i+1}) - e_{i+1}] (1 + i_{t+1}) - \left[ \left( b^{(1)}_{t+1} + s^{(1)}_{t+1} \right) q_{x+t}^{(1)} \left( b^{(2)}_{t+1} + s^{(2)}_{t+1} \right) \left( 1 - q^{(1)}_{x+t} \right) q^{(2)}_{x+t} + p^{(2)}_{x+t} \right] t + 1V \] \hspace{1cm} (14.42)

**ACTEX MLC Chapter 9 Study Manual Vol II Multiple Decrement Models: Applications**

**Chapter 15 MQR Models with Variable Interest Rates**

**15.4 Forward Interest Rates**
\[ (1 + y_5)^5 = (1 + y_1)^1 \cdot (1 + f_{1,4})^4 \] \hspace{1cm} (15.1)
\[ (1 + y_5)^6 = (1 + y_1)^2 \cdot (1 + f_{2,2})^2 \] \hspace{1cm} (15.2)
\[ (1 + y_5)^7 = (1 + y_1)^3 \cdot (1 + f_{3,5})^5 = (1 + y_5)^{k+5-k} = (1 + y_5)^5 \] \hspace{1cm} (15.3)
\[ (1 + y_5)^8 = (1 + f_{1,1}) (1 + f_{0,0}) \]

**Chapter 12 ACTEX MLC Study Manual Vol II Interest Rate Risk**

**Spot interest rate** \( v(t) = (1 + y_t)^{-t} \)

**Forward interest rate** \( (1 + f_{i,k})^k = \frac{(1 + y_{t+k})^{t+k}}{(1 + y_t)^t} = \frac{v(t)}{v(t+k)} \)

**Chapter 16 MQR Universal Life Insurance**

**16.2 Indexed Universal Life Insurance.**

a) Point-to-point method: \( i_P = \frac{\text{Final Index Closing Value}}{\text{Initial Index Closing Value}} - 1 \), \hspace{1cm} (16.1)

b) Monthly average method: \( i_{MA} = \frac{\frac{1}{12} \sum \text{Monthly Index Closing Values}}{\text{Initial Index Closing Value}} - 1 \). \hspace{1cm} (16.2)

**16.3 Pricing Considerations**

**Mortality rate, Lapse rate, Expenses, Investment Income.**

Double decrement model: \( p_{x}^{(r)} = 1 - q_{x}^{(r)} = 1 - q_{x}^{(d)} - q_{x}^{(w)} \). \hspace{1cm} (16.3)

Withdrawal at end of year only: \( p_{x}^{(r)} = \left( 1 - q_{x}^{(d)} \right) \left( 1 - q_{x}^{(w)} \right) \). \hspace{1cm} (16.4)


**16.4 Reserving Considerations**

**Definitions:**

\( ULI = \) Universal Life Insurance. Policy is marked by (a) extensive policyholder choice, (b) policyholder participation in interest rate risk, and (c) secondary guarantee features of coverage.

\( VUL = \) Variable Universal Life insurance. Separate investment accounts for net contributions.

\( EIUL = \) Equity-Indexed Universal Life insurance. Interest/investment is credited to contract at rate that depends on some published stock index such as SP500, DJIA, or EAFE index.

\( SC_t = \) Surrender Charge at time \( t \).

\( M&E = \) Mortality and Expenses Charge at time \( t \).

\( NAR_t = \) Net Amount at Risk at time \( t \).

\( AV_t = \) Account Value at time \( t \).

\( CV_t = \) Cash Value at time \( t \). \( (CV_t = AV_t - SC_t) \)

\( NAIC = \) National Association of Insurance Commissioners

Policy Guarantees = Guarantees in a policy.

Roll Forward process = A process to bring forward to the future a financial value.

\( GMP = \) Guaranteed Maturity Premium. Level gross premium sufficient to endow the policy at its maturity date. based on the policy guarantees of premium loads, interest rates, and expense and mortality charges.

\( GMF = \) Guaranteed Maturity Benefits. Calculated based on the roll forward of the GMP and the policy guarantees.

\( GDB = \) Guaranteed Death Benefits.

\( GMB = \) Guaranteed Maturity Benefits.
$PVFB_t = $Present Value$ at time t of the projected Future Benefits. 
$PVFP_t = $Present Value$ at time t of the Future GMP stream. 
CRVM = Commissioner’s reserve valuation method 
$CSV_t = $Cash Surrender Value$ at time t. 
AMR = $Alternative Minimum Reserves$. 

16.4.1 Basic Universal Life (ULI) Process for 1983 NAIC regulation to define a minimum reserving standard for UL products. 

a) At policy issue, 
1. a guaranteed maturity premium ($GMP$) is calculated as the level gross premium sufficient to endow the policy at its maturity date. The GMP is based on the policy guarantees of premium loads, interest rates, and expense and mortality charges. 
\[GMP_0 = \text{policy guarantees of } f(\text{premium loads, } i, M&E).\]
2. a sequence of guaranteed maturity funds (GMF) is calculated based on the roll forward of the GMP and the policy guarantees 
\[\text{a sequence } GMF = \text{roll forward of } f(GMP, \text{policy guarantees}).\]

b) At the valuation date, $t$, 
3. actual $AV_t$ determined by the account value roll forward process. 
4. the ratio of the actual account value to the GMF is calculated as 
\[r_t = \frac{AV_t}{GMF_t}, \quad r_t \leq 1 \quad (16.5)\]
5. $\max(AV_t, GMF_t)$ is projected forward based on the GMP and the policy guarantees. This produces a sequence of GDB and GMB. 
6. $PVFB_t$ and $PVFP_t$ are calculated using valuation assumptions. Then the pre-floor CRVM reserve is defined as 
\[tV_{\text{pre-floor CRVM}} = r_t(PVFB_t - PVFP_t) \quad (16.6)\] 
with $r_t$ as defined above. 
7. $tV_{\text{floor CRVM}} = \max(\frac{1}{2}$-month term reserve based on minimum valuation mortality and interest, $CSV_t)$. 
8. $tV_{\text{final CRVM}} = \max(tV_{\text{pre-floor CRVM}}, tV_{\text{floor CRVM}})$. 

The regulation also defines alternative minimum reserves (AMR). 
1. the valuation net premium is calculated at policy issue ($t = 0$) based on the GMP and the policy guarantees. 
2. If the GMP < the valuation net premium (VNP), 
the reserve held = $\max(a, b)$ 
where $a =$ the reserve calculated using the actual method and assumptions of the policy + VNP, 
$b =$ the reserve calculated using the actual method but with minimum valuation assumptions + GMP. 

16.4.1 Indexed Universal Life (eIUL) 
NAIC Actuarial Guideline 36 (AG 36) specifies the valuation standards for IUL contracts. 3 computational methods: 
1) The implied guaranteed rate (IGR) method: which requires insurers to satisfy the hedged-as-required criteria. These criteria set forth a strenuous constraint requiring exact, or nearly exact, hedging, as well as an indexed interest-crediting term of not more than one year. 
2) The CRVM with updated market value (CRVM/UMV) method: 
must be used if the contract has an indexed interest-crediting term of more than one year, or if the renewal participation rate guarantee gives an implied guaranteed rate greater than the maximum valuation rate. This method can be volatile when market conditions change. 
3) The CRVM with updated average market value (CRVM/UAMV) method: 
is a hybrid of the other two, designed for an insurer who qualifies for the first method above but does not wish to satisfy the hedged-as-required criteria. 

The CRVM/UMV method has calculation steps as follows: 
a) The issue date ($t = 0$) calculations are as follows:
1. An implied guaranteed interest rate (IGR) for the duration of the initial term, is the guaranteed rate plus the accumulated option cost expressed as a percentage of the policy value to which the indexed benefit is applied. In turn, the accumulated option cost is the amount needed to provide the index-based benefit in excess of any other interest rate guarantee, accumulated to the end of the initial term at the appropriate maximum valuation rate.

2. An implied guaranteed rate for the period after the initial term.

3. The GMP, GMF, and valuation net premium based on the implied guaranteed rate.

b) The valuation date \((t = t)\) calculations are as follows:

1. The implied guaranteed rate for the remainder of the current period, using the option cost based on the market conditions at the valuation date.

2. The implied guaranteed rate for the period following the current period, based on the option cost on the valuation date.

3. A re-projection of future guaranteed benefits based on the implied valuation date.

4. The present value of the re-projected future guaranteed benefits.

Note that the GMP, GMF, and valuation net premium remain the same as calculated at issue \((t = 0)\).

16.4.4 Contracts with Secondary Guarantees

NAIC Actuarial Guideline 38 (AG 38) for reserves for UL products with secondary guarantees have 9 steps as follows:

1. The minimum gross premium required to satisfy the secondary guarantees is derived at issue \((t = 0)\) of the contract; the value of this premium will depend on whether the stipulated premium or the shadow fund method is in use. Its calculation uses the policy charges and credited interest rate guaranteed in the contract.

2. The basic and deficiency reserves for the secondary guarantees are calculated using the minimum gross premium described in Step (1).

3. The amount of actual contributions made in excess of the minimum gross premiums is determined, again with the process depending on whether the stipulated premium method or the shadow fund method is used.

4. At the valuation date, \(t\), a determination is made regarding amounts needed to fully fund the secondary guarantee.

   (a) Under the shadow fund method, this would be the amount of the shadow fund account needed to fully fund the guarantee.

   (b) Under contracts not using the shadow fund method, this would be the amount of cumulative premiums paid in excess of the required level such that no future premiums are required to fully fund the guarantee.

Special rules apply to policies for which the secondary guarantee cannot be fully funded in advance. Here a prefunding ratio, \(r\), \((r \leq 1)\), is calculated that measures the level of prefunding for the secondary guarantee, and is eventually used in the calculation of reserves. It is defined as

\[
r = \frac{\text{Excess Payment}}{\text{Net Single Premium Required to Fully Fund the Guarantee}}.
\]

5. At the valuation date, \(t\), the net single premium for the secondary guarantee coverage for the remainder of the secondary guarantee period is computed. \(NSP_t\).

6. A net amount of additional premiums is determined by multiplying the prefunding ratio described in Step (4) times the difference between the net single premium of Step (5) and the basic plus deficiency (if any) reserve of Step (2). \(r(NSP_t - bpd V)\)

7. A reduced deficiency reserve is determined by multiplying the deficiency reserve (if any) by the complement of the pre-funding ratio from Step (4). \((1 - r)dV\)
8. Then the actual reserve is the lesser of (a) the net single premium of Step (5), or (b) the amount in Step (6) plus the basic and deficiency (if any) reserve from Step (2). This result might be reduced by applicable policy surrender charges. $V = \min(NSP_t, Step6 + Step2)$

9. An increased basic reserve is computed by subtracting the reduced deficiency reserve of Step (7) from the reserve computed in Step (8), which then becomes the basic reserve. $bV = Step8 - Step7$.

ACTEX MLC Chapter 14 Study Manual Vol II Universal Life Insurance

14.1 Basic Policy Design

Account Value Accumulation $A_{V_t} = (A_{V_{t-1}} + P_t - EC_t - Col_t) (1 + i_t^f)$

14.2 Cost of Insurance and Surrender Value

Total Death Benefit

Specified Amount (Type A): max $(FA, \gamma_t A_{V_t})$
Specified Amount plus the Account Value (Type B): max $(A_{V_t} + X, \gamma_t A_{V_t})$

Additional Death Benefit

Specified Amount (Type A): $ADB_t = \max (FA - A_{V_t}, (\gamma_t - 1) A_{V_t})$
Specified Amount plus the Account Value (Type B): $ADB_t = \max (X, (\gamma_t - 1) A_{V_t})$

General Formula for the cost of Insurance $CoI_t = q_t^* v_q (FA - (A_{V_{t-1}} + P - EC_t) (1 + i_t^c))$

where $CoI_t$ is the cost of insurance for the $t^{th}$ time period, deducted from the account value at time $t - 1$, $q_t^*$ is the death probability (for the $t^{th}$ time period) used to calculate the cost of insurance, $v_q$ is the discount factor for discounting the cost of insurance to time $t - 1$, $ADB_t$ is the additional death benefit at time $t$.

Cost of Insurance for a Specified Amount (Type A) Policy

$CoI_t = \max \left( CoI_t^f, CoI_t^c \right)$

where $CoI_t^f = \frac{q_t^* v_q (FA - (A_{V_{t-1}} + P - EC_t) (1 + i_t^c))}{1 - q_t^* v_q (1 + i_t^c)}$

$CoI_t = \frac{q_t^* v_q (1 + i_t^c) (\gamma_t - 1) (A_{V_{t-1}} + P - EC_t)}{1 + q_t^* v_q (1 + i_t^c) (\gamma_t - 1)}$

Cost of Insurance for a Specified Amount plus the Account Value (Type B) Policy

$CoI_t = \max \left( CoI_t^f, CoI_t^c \right)$

where $CoI_t^f = q_t^* v_q X$ and $CoI_t^c = \frac{q_t^* v_q (1 + i_t^c) (\gamma_t - 1) (A_{V_{t-1}} + P - EC_t)}{1 + q_t^* v_q (1 + i_t^c) (\gamma_t - 1)}$

14.5 Profit Testing

Expected Profit: $Pr_t = (A_{V_{t-1}} + P - EC_t) (1 + i_t^c) - EDB_t - ESB_t - EA_{V_t}$

Chapter 17 MQR Deferred Variable Annuities

17.2 Deferred Annuity Products

17.2.2 Variable Deferred Annuity

Investment Advisory Fee: $IADF_t(n) = FV_{t-1}(n) \cdot \left( (1 + IAF_t(n))^{1/365} - 1 \right)$, (17.1)

Net Investment Rate for day $t$: $NIR_t(n) = \frac{Nh_t(n) - IAF_t(n) + RCG_t(n) + UCG_t(n)}{FV_{t-1}(n)}$, (17.2)

Net Investment Factor: $NIR_t(n) = 1 + NIR_t(n)$, (17.3)

Sub-account n Fund Value recursion: $FV_t(n) = FV_{t-1}(n) \cdot NIF_t(n) - EXP_t(n)$, (17.4)

Overall contract Account Value on day $t$: $A_{V_t} = \sum_n FV_t(n)$, (17.5)

17.2.3 Equity -Indexed Deferred Annuity

a) point-to-point: $i_P = \frac{\text{Index value on closing day of index period}}{\text{Index value on initial day of index period}} - 1$, (17.6)

b) monthly ave: $i_{MA} = \frac{1}{12n} \left[ \text{Sum of index values on the last day of each month during index period} \right]$ - 1. (17.7)

c) with ratcheting $i_P = \frac{\text{Index value on closing day } t \text{ of index period}}{\text{Index value on day } t - 1 \text{ of index period}} - 1$,

d) with ratcheting $i_{MA} = \frac{1}{12n} \left[ \text{Sum of index values on the last day of each month during index period} \right]$ - 1.
17.3 Immediate Annuity Products

17.3.2 Variable Immediate Annuity

\[ APU = \frac{P_t}{PUV_t} \quad (17.8) \]

\[ PUV_t = PUV_{t-1} \left( \frac{NIF_t}{1 + AIR} \right), \quad (17.9) \]

\[ P_t = (APU) (PUV_{t-1}) \left( \frac{NIF_t}{1 + AIR} \right) = P_{t-1} \left( \frac{NIF_t}{1 + AIR} \right). \quad (17.11) \]

ACTEX MLC Chapter 13 Profit Testing

Profit vector \( \left[ Pr_0, Pr_1, Pr_2, Pr_3, \ldots \right]' \)

Profit signature \( \Pi = (\Pi_0, \Pi_1, \Pi_2, \Pi_3, \ldots)' = (Pr_0, Pr_1, px Pr_2, 2px Pr_3, \ldots)' \)

Expected profit that emerges in \((h + 1)\)th year

\[ Pr_{h+1} = N[(h V^g + G_h(1 - c_h) - e_h)(1 + i) - (b_{h+1} + E_{h+1})q_{x+h} + p_{x+h} h_{1+1} V^g] \]

\[ = N[(G_h(1 - c_h) - e_h)(1 + i) - (b_{h+1} + E_{h+1})q_{x+h} + (1 + i) V^g - p_{x+h} h_{1+1} V^g] \quad (1) \]

\( Pr_1 = (1) \) without acquisition cost. \( Pr_0 = 0 \) all acquisition costs

Extension to Multiple State Models (assuming \( N = 1 \))

\[ Pr^{(j)}_{k+h} = [h V^{(j)} + G_h^{(j)}(1 - c_h^{(j)}) - e_h^{(j)}](1 + i) - \sum_{k=0}^{n} (b_{k+h+1}^{(j)} + E_{k+h+1}^{(j)}) p_{k+h}^{(j)} \]

\[ \Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots)' \]

\[ \Pi_t = \sum_{k=0}^{n} t-1 p_{x}^{(j)} Pr^{(j)}_{t} \quad \Pi_0 = Pr^{(0)}_{0}, \]

\[ \Pi_1 = \sum_{k=0}^{n} 0 p_{x}^{(j)} Pr^{(j)}_{t} = 0 \]

Traditional Insurance Policies with Withdrawal (assuming \( N = 1 \))

\[ Pr^{(0)}_{k+h} = [v V^{(0)} + G_h(1 - c_h) - e_h](1 + i) - h_{1+1} V^{(0)} + h_{1} V_{x+h}^{(1)} + (b_{h+1} + E_{h+1})q_{x+h}] \quad h \geq 0 \]

\[ Pr^{(1)}_{h+1} = 0, \quad Pr^{(2)}_{h+1} = 0, \quad h \geq 1 \]

\( Pr^{(0)}_{t} = - \) acquisition costs

\[ \Pi_{h} = h_{1} p_{x}^{(r)} Pr^{(0)}_{h}, \quad h \geq 2 \]

\[ \Pi_{1} = Pr^{(1)}_{1}, \quad \Pi_{0} = Pr^{(0)}_{0} \]

Extension to Policies with Continuous Benefit (assuming \( N = 1 \))

\[ Pr_{h+1} = [h V^g + G_h(1 - c_h) - e_h(1 + i) - (1 + i)^{1/2}(b_{h+0.5} + E_{h+0.5})q_{x+h} + p_{x+h} h_{1+1} V^g] \]

13.2 Profit Measures

1. Net present value (NPV):

\[ NPV(r) = \sum_{k=0}^{n} C_k \frac{1}{(1 + r)^k}. \]

If the NPV is positive, then the investment is deemed profitable.

2. Internal rate of return (IRR): The internal rate of return is the zero of the equation

\[ NPV(r) = \sum_{k=0}^{n} C_k \frac{1}{(1 + r)^k} = 0. \]

3. Discounted payback period (DPP)

Given the hurdle rate \( r \), the discounted payback period (also known as the break-even period) is the smallest value of \( m \) such that \( \sum_{k=0}^{m} C_k \frac{1}{(1 + r)^k} \geq 0 \).

DPP is the time until the investment starts to make a profit.

4. Profit margin

Profit margin is the NPV of the net cash flows as a percentage of the NPV of the revenues. Suppose that the revenue cash flows are \( R_0, R_1, R_2, \ldots R_n \). Then the profit margin is

\[ \frac{NPV(r)}{\sum_{k=0}^{n} R_k} \]

For life insurance, NPV is expected present value of the profits at issue and premiums as revenues (mortality rate taken into account).

\[ \text{Profit margin} = \frac{\sum_{k=0}^{n} \Pi_k}{\sum_{k=0}^{n} G_{k, k} p_x} \frac{1}{\sum_{k=0}^{n} \frac{1}{(1 + r)^k}}. \]

5. NPV as a proportion of the acquisition costs

\[ \text{NPV as a proportion of the acquisition costs} = \frac{1}{\Pi_0} \sum_{k=0}^{n} \frac{\Pi_k}{(1 + r)^k}. \]