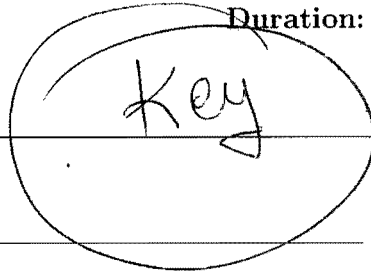


King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Exam I - Term 131

Duration: 120 minutes



Name: _____ ID Number: _____

Section Number: _____ Serial Number: _____

Class Time: _____ Instructor's Name: _____

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 8 Problems)

| | Points | Maximum Points |
|--------|--------|----------------|
| page 1 | | 13 |
| page 2 | | 13 |
| page 3 | | 12 |
| page 4 | | 16 |
| page 5 | | 12 |
| page 6 | | 12 |
| page 7 | | 8 |
| page 8 | | 14 |
| Total | | 100 |

1. A) (7 points) Convert the parametric equations

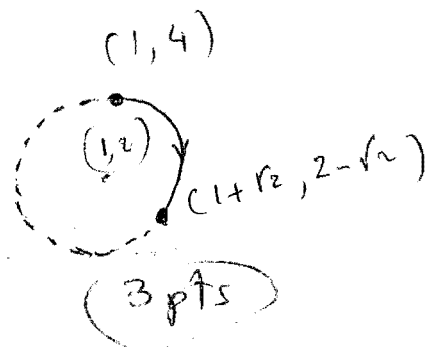
$$x = 1 + 2 \sin t, \quad y = 2 \cos t + 2, \quad 0 \leq t \leq \frac{3\pi}{4}$$

into cartesian (rectangular) equation. Sketch the curve with the direction of motion.

We have $2 \sin t = x - 1$ and $2 \cos t = y - 2$. This implies

$$(x-1)^2 + (y-2)^2 = 4(\sin^2 t + \cos^2 t) = 4 \Rightarrow$$

(4 pts) $(x-1)^2 + (y-2)^2 = 4$. With $0 \leq t \leq \frac{3\pi}{4}$, the curve is the portion of the circle clockwise from the point $(1, 4)$ to the point $(1+\sqrt{2}, 2-\sqrt{2})$ circle centered at $(1, 2)$ with radius 2



B) (6 points) Determine the concavity of the curve given by

$$x = \theta - \sin \theta, \quad y = 1 - \cos \theta, \quad \text{at } \theta = \pi.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin \theta}{1 - \cos \theta} \quad (1 \text{ pt}) = \frac{\cos \theta - 1}{1 - \cos \theta}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta} \left[\frac{dy}{dx} \right]}{\frac{dx}{d\theta}} = \frac{\frac{\cos \theta (1 - \cos \theta) - \sin \theta}{(1 - \cos \theta)^2}}{1 - \cos \theta} \quad (2 \text{ pts}) = \frac{-1}{(1 - \cos \theta)^2} \quad (2 \text{ pts})$$

at $\theta = \pi$ we have

$$\left. \frac{d^2y}{dx^2} \right|_{\theta = \pi} = \frac{(-1)(2) - 0}{8} = -\frac{2}{8} = -\frac{1}{4} < 0$$

The curve is concave down at $\theta = \pi$ (4 pt)

2. A) (6 points) Find an equation of the line that passes through the point (1, 1) and parallel to the tangent line to the curve

$$x = \ln t \quad y = \sqrt{t+1} \quad \text{at } t = 3.$$

$$\text{slope} = \frac{dy}{dx} \Big|_{t=3} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2\sqrt{t+1}}}{\frac{1}{t}} = \frac{t}{2\sqrt{t+1}} \quad (2 \text{ pts})$$

$$\text{slope at } t=3 = \frac{3}{4} \quad (1 \text{ pt})$$

use now the point-slope form, the equation of the line is given by

$$y = \frac{3}{4}(x-1) + 1 \quad (2 \text{ pts})$$

$$= \frac{3}{4}x + \frac{1}{4}$$

- B) (7 points) Let C be the portion of the curve $x = 3 \cos \theta$, $y = 3 \sin \theta$ from (3, 0) to $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$. Find the area of the surface obtained by rotating C about the x -axis.

first observe that (3, 0) \rightarrow $t = 0$ and $(\frac{3}{2}, \frac{3\sqrt{3}}{2}) \rightarrow t = \frac{\pi}{3}$ (2 pts)

$$S = 2\pi \int_0^{\frac{\pi}{3}} 3 \sin t \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} dt \quad (3 \text{ pts})$$

$$= 6\pi \int_0^{\frac{\pi}{3}} \sin t \sqrt{9(\sin^2 t + \cos^2 t)} dt$$

$$= 18\pi \int_0^{\frac{\pi}{3}} \sin t dt = -18\pi \left(\cos t \Big|_0^{\frac{\pi}{3}} \right) \quad (1 \text{ pt})$$

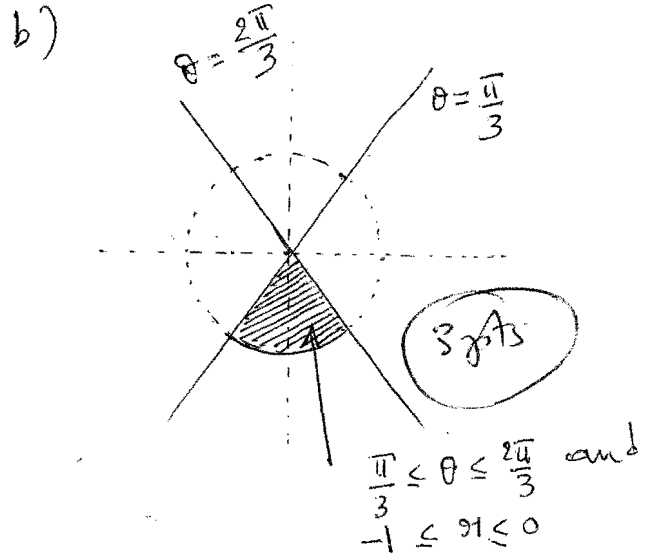
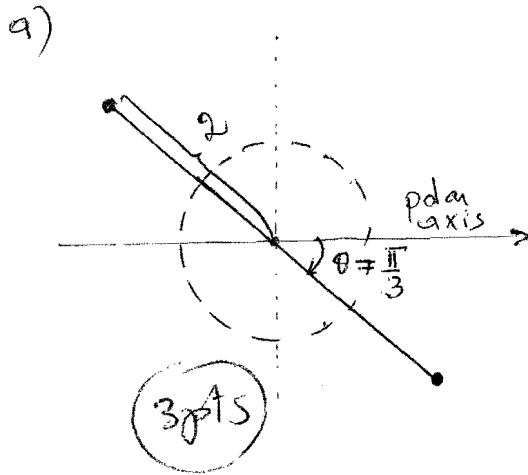
$$= -18\pi \left(\frac{1}{2} - 1 \right)$$

$$= 9\pi \quad (4 \text{ pts})$$

3. A)(6 points) Graph the set of points (r, θ) whose polar coordinates satisfy the given conditions:

a) $\theta = -\frac{\pi}{3}$ and $-2 \leq r \leq 2$

b) $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$ and $-1 \leq r \leq 0$



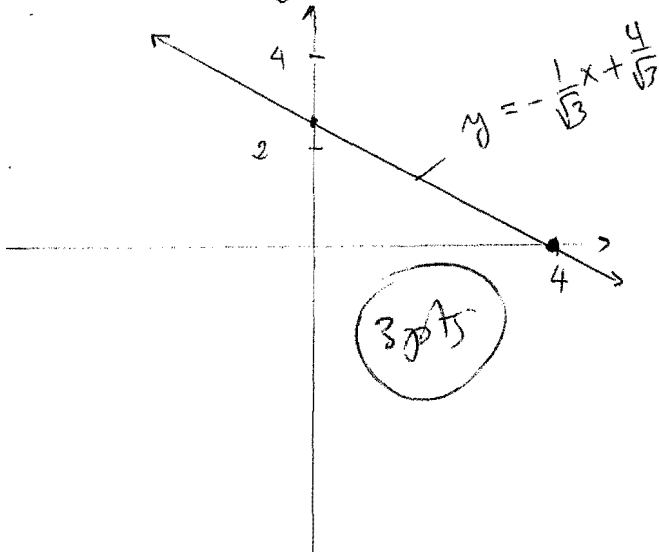
B) (6 points) Sketch the polar curves given by

a) $r \sin\left(\theta + \frac{\pi}{6}\right) = 2$

b) $r = \cot \theta \csc \theta$

a) $r \sin\left(\theta + \frac{\pi}{6}\right) = 2 \iff$
 $r \sin \theta \cos \frac{\pi}{6} + r \cos \theta \sin \frac{\pi}{6} = 2$
 $\iff \frac{\sqrt{3}}{2} y + \frac{x}{2} = 2 \iff$

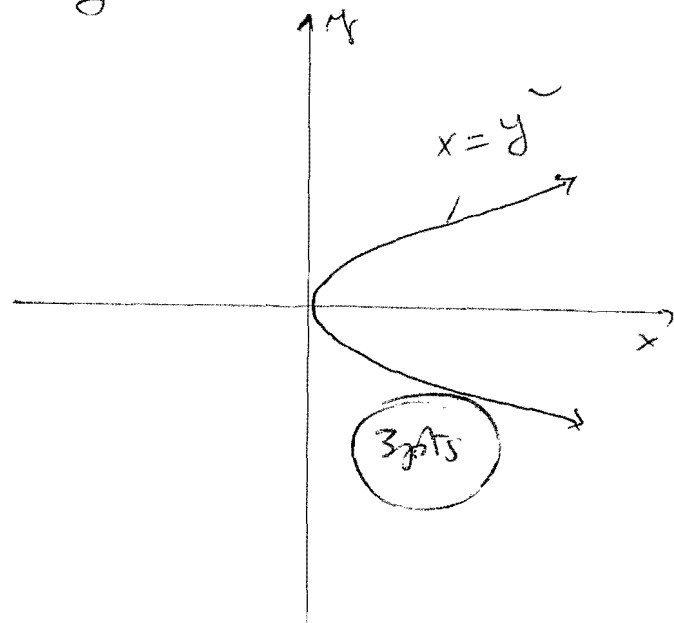
$y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}$



b) $r = \cot \theta \csc \theta = \frac{\cos \theta}{\sin^2 \theta}$

$\iff r^2 \sin^2 \theta = r \cos \theta \quad (=)$

$y^2 = x$



4. A) (8 points) Consider the polar curve C given by $r = 1 - \cos\theta$, $0 \leq \theta \leq 2\pi$. Find the slope of C at $\theta = \frac{\pi}{2}$ and the total length of C .

$$\frac{dy}{dx} \Big|_{\theta = \frac{\pi}{2}} = \frac{\frac{dr}{d\theta} \sin\theta + r \cos\theta}{\frac{dr}{d\theta} \cos\theta - r \sin\theta} \Big|_{\theta = \frac{\pi}{2}} = \frac{(1)(2) + 0}{(1)(0) - 2(1)} = -1$$

(4 pts) (2 pts)

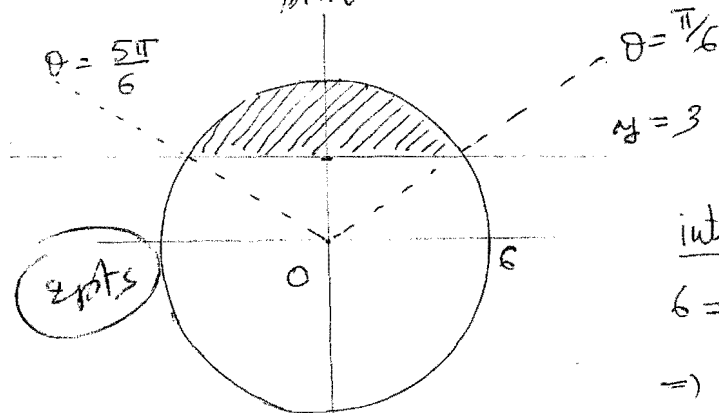
$$L = \int_0^{2\pi} \sqrt{(1 - \cos\theta)^2 + \sin^2\theta} d\theta = \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos\theta} d\theta =$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{2 \sin^2\left(\frac{\theta}{2}\right)} d\theta = 2 \int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 8$$

(3 pts) (4 pts)

- B) (8 points) Find the area inside the circle $r = 6$ and above the line $r = 3 \csc\theta$.

$$r = 3 \csc\theta \Leftrightarrow r = \frac{3}{\sin\theta} \Leftrightarrow r \sin\theta = 3 \Leftrightarrow y = 3$$



intersecting points

$$6 = \frac{3}{\sin\theta} \Leftrightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

(2 pts)

$$S = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{6^2}{2} d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{9 \csc^2\theta}{2} d\theta$$

(2 pts)

$$= 18 \left(\frac{5\pi}{6} - \frac{\pi}{6} \right) + \frac{9}{2} \left(\cot\theta \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \right)$$

(4 pts)

$$= 12\pi - 9\sqrt{3}$$

(4 pts)

5. A) (6 points) Give an equation of the circle of radius 2 centered at $(0, 2, 0)$ and lying in a plane parallel to the yz -plane.

$$(y-2)^2 + z^2 = 4, \quad x = 0$$

3 pts

2 pts

- B) (6 points) Write inequalities to describe the solid rectangular box in the first octant bounded by the planes $x = 4$, $y = 7$ and $z = 8$

$$0 \leq x \leq 4, \quad 0 \leq y \leq 7, \quad 0 \leq z \leq 8$$

2 pts

2 pts

$$0 \leq z \leq 8$$

2 pts

6. A) (6 points) Find the component form of the vector \vec{v} making an angle of $\frac{\pi}{6}$ with the positive x -axis with $|\vec{v}| = 2$.

$$\vec{v} = \left\langle |\vec{v}| \cos\left(\frac{\pi}{6}\right), |\vec{v}| \sin\left(\frac{\pi}{6}\right) \right\rangle$$

$$= \left\langle 2 \cdot \frac{\sqrt{3}}{2}, 2 \cdot \frac{1}{2} \right\rangle$$

$$= \langle \sqrt{3}, 1 \rangle = \sqrt{3}\vec{i} + \vec{j}$$

3 pts

2 pts

1 pt

- B) (6 points) Find the angle between the vectors $\vec{u} = \langle 1, -2, 3 \rangle$ and $\vec{v} = \langle -2, 3, 1 \rangle$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

1 pt

$$\vec{u} \cdot \vec{v} = (1)(-2) + (-2)(3) + (3)(1) = -5$$

1 pt

$$|\vec{u}| = \sqrt{14}$$

4 pts

$$|\vec{v}| = \sqrt{14}$$

4 pts

so

$$\cos\theta = \frac{-5}{14}$$

2 pts

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-5}{14}\right)$$

7. A) (8 points) If the angle between \vec{u} and \vec{v} is $\theta = \frac{\pi}{3}$ and that $|\vec{u}| = 3$, $|\vec{v}| = 2$, find $|\vec{u} + \vec{v}|$.

$$\begin{aligned} |\vec{u} + \vec{v}|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) && \text{2pts} \\ &= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} && \text{2pts} \\ &= |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 && \text{1pt} \\ &= |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}|\cos\frac{\pi}{3} && \text{2pts} \\ &= 9 + 4 + 2(3)(2)\left(\frac{1}{2}\right) \\ &= 19 && \text{1pt} \end{aligned}$$

8. A) (6 points) Determine whether the vectors $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$ and $\vec{w} = \langle 0, -9, 18 \rangle$ lie on the same plane.

$$V = \left| \vec{u} \cdot (\vec{v} \times \vec{w}) \right| = 0 \quad (4 \text{ pts})$$

We conclude that the three vectors are on the same plane. (2 pts)

- B) (8 points) Let $\vec{u} = \langle 1, -1, 2 \rangle$, and $\vec{v} = \langle 3, -2, 0 \rangle$. If $\vec{a} = \text{proj}_{\vec{v}} \vec{u}$,

find $w = (2\vec{u} - \vec{a}) \cdot \frac{\vec{v}}{3}$

We know that $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to \vec{v} so $\vec{u} - \vec{a}$ is orthogonal to \vec{v} .

$$\begin{aligned} w &= (\vec{u} + \vec{u} - \vec{a}) \cdot \frac{\vec{v}}{3} && (2 \text{ pts}) \\ &= \vec{u} \cdot \frac{\vec{v}}{3} + (\vec{u} - \vec{a}) \cdot \frac{\vec{v}}{3} && (1 \text{ pt}) \\ &= \frac{1}{3} \vec{u} \cdot \vec{v} + 0 && (4 \text{ pts}) \\ &= \frac{1}{3} (3 + 2 + 0) = \frac{5}{3} && (1 \text{ pt}) \end{aligned}$$