1. (8 points) Let $y = c_1 \cos\left(\frac{1}{\alpha}x\right) + c_2 \sin\left(\frac{1}{\alpha}x\right)$, $\alpha \neq 0$(constant), be a 2-parameter family of solutions of the differential equation $y'' + \frac{1}{\alpha^2}y = 0$. Determine whether a member of the family can be found that satisfies the boundary conditions $y\left(\frac{\alpha\pi}{2}\right) = 1$ and $y'(0) = 0$.

2. (8 points) Without using the Wronskian, determine whether the set of functions

$$f_1(x) = \sqrt{x} + 3, \;\; f_2(x) = \sqrt{x} + 3x, \;\; f_3(x) = x - 1,$$

is linearly independent on the interval $(0, \infty)$.

3. (a)(4 points) Verify that $y_{p_1} = xe^{-x}$ and $y_{p_2} = x^2 - 8x + 23$ are, respectively, particular solutions of

$$y'' + 3y' + y = (-x+1)e^{-x} \text{ and } y'' + 3y' + y = x^2 - 2x + 1$$

(b) (6 points) Use part (a) to find a particular solution of

$$y'' + 3y' + y = (2x-2)e^{-x} + 3(x-1)^2$$

4. (12 points) Given that $y_1 = x$ is a solution of the differenital equation

$$(1 - x^2)y'' + 2xy' - 2y = 0 \text{ on } (-1, 1),$$

find a second solution $y_2(x)$ that is linearly independent of y_1.

5. (12 points) Solve the following boundary value problem

$$y''' + 4y' = 0, \qquad y(0) = 0,\ y'(0) = 1, \qquad y\left(\frac{\pi}{2}\right) = 2.$$

6. (13 points) Solve the given differential equation by undetermined coefficients

$$y'' + 2y' + 2y = 5e^{6x}$$

7. (13 points) Solve $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 4e^{3x}\ln x, \quad (x > 0).$

8. a) (8 points) Solve $xy'' - 5y' = 0$

b) (8 points) Use the substitution $x = e^t$ to solve the non-homogeneous differential equation

$$x\frac{dy}{dx} + y = \ln x.$$

9. Find a linear differential operator of lowest order that annihilates the function:

   a) (4 points) $(3 - e^x)^2 + 6x$

   b) (4 points) $\sin^2(4x)$