1. (8 points) Let \( y = c_1 \cos \left( \frac{1}{\alpha} x \right) + c_2 \sin \left( \frac{1}{\alpha} x \right) \), \( \alpha \neq 0 \) (constant), be a 2-parameter family of solutions of the differential equation \( y'' + \frac{1}{\alpha^2} y = 0 \). Determine whether a member of the family can be found that satisfies the boundary conditions \( y \left( \frac{\alpha \pi}{2} \right) = 1 \) and \( y'(0) = 0 \).
2. (8 points) Without using the Wronskian, determine whether the set of functions

\[ f_1(x) = \sqrt{x} + 3, \quad f_2(x) = \sqrt{x} + 3x, \quad f_3(x) = x - 1, \]

is linearly independent on the interval \((0, \infty)\).
3. (a) (4 points) Verify that $y_{p_1} = xe^{-x}$ and $y_{p_2} = x^2 - 8x + 23$ are, respectively, particular solutions of

$$y'' + 3y' + y = (-x + 1)e^{-x} \text{ and } y'' + 3y' + y = x^2 - 2x + 1$$

(b) (6 points) Use part (a) to find a particular solution of

$$y'' + 3y' + y = (2x - 2)e^{-x} + 3(x - 1)^2$$
4. (12 points) Given that $y_1 = x$ is a solution of the differential equation

$$(1 - x^2)y'' + 2xy' - 2y = 0 \text{ on } (-1, 1),$$

find a second solution $y_2(x)$ that is linearly independent of $y_1$. 
5. (12 points) Solve the following boundary value problem

\[ y''' + 4y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y\left(\frac{\pi}{2}\right) = 2. \]
6. (13 points) Solve the given differential equation by undetermined coefficients

\[ y'' + 2y' + 2y = 5e^{6x} \]
7. (13 points) Solve \( \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = 4e^{3x} \ln x, \ (x > 0). \)
b) (8 points) Use the substitution \( x = e^t \) to solve the non-homogeneous differential equation

\[
x \frac{dy}{dx} + y = \ln x.
\]
9. Find a linear differential operator of lowest order that annihilates the function:

a) (4 points) \((3 - e^x)^2 + 6x\)

b) (4 points) \(\sin^2(4x)\)