MATH 202.6 (Term 131)

Quiz 6 (Sects. 8.1, 8.2) Duration: 20mn

Name: ID number:

Solve the homogeneous linear system

$$X' = \begin{pmatrix} 0 & -1 & -4/3 \\ 5/3 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} X$$

$X' = AX$

$\det(A - \lambda I) = 0 \iff$

$$\begin{vmatrix} -\lambda & -1 & -4/3 \\ \frac{5}{3} & 2-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$-\lambda(2-\lambda)(-1-\lambda) - \frac{5}{3}(1+\lambda) + \frac{4}{3}(2-\lambda) = 0$

$-\lambda^3 + \lambda^2 - \lambda + 1 = 0$

$-(\lambda^2+1)(\lambda-1) = 0 \Rightarrow \lambda = 1, \ \lambda = \pm i$

Case 1 : $\lambda = 1$

$$\left(\begin{array}{ccc|c} -1 & -1 & -4/3 & 0 \\ 5/3 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 0 \\ 0 & -\frac{2}{3} & -\frac{20}{9} & 0 \\ 0 & -1 & -\frac{10}{3} & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 0 \\ 0 & 1 & \frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$K_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x - 2z = 0 \\ y + \frac{10}{3} z = 0 \end{cases} \Rightarrow \begin{cases} x = 2z \\ y = -\frac{10}{3} z \end{cases}$

$\Rightarrow K_1 \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix} \Rightarrow X_1 = \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix} e^t$

Case 2 $\lambda = i$

$$\left(\begin{array}{ccc|c} -i & -1 & -4/3 & 0 \\ \frac{5}{3} & 2-i & 0 & 0 \\ 1 & 0 & -1-i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1-i & 0 \\ 0 & -1 & -\frac{1}{3}-i & 0 \\ 0 & 2-i & \frac{5}{3}(1+i) & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1-i & 0 \\ 0 & 1 & \frac{1}{3}+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad K_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$$

$\begin{cases} x - z(1+i) = 0 \\ y + z(\frac{1}{3}+i) = 0 \end{cases} \Rightarrow \begin{cases} x = z(1+i) \\ y = -z(\frac{1}{3}+i) \end{cases}$

$K_2 \begin{pmatrix} 1+i \\ -\frac{1}{3}-i \\ 1 \end{pmatrix}$

$B_1 = \mathrm{Re}\, K_2 = \begin{pmatrix} 1 \\ -1/3 \\ 1 \end{pmatrix}, \quad B_2 = \mathrm{Im}\, K_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$X_2 = B_1 \cos t - B_2 \sin t$

$= \begin{pmatrix} 1 \\ -\frac{1}{3} \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \sin t$

$X_3 = B_2 \cos t + B_1 \sin t$

$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -\frac{1}{3} \\ 1 \end{pmatrix} \sin t$

The general solution of the DE is $X = c_1 X_1 + c_2 X_2 + c_3 X_3$.

MATH 202.9 (Term 131)

Quiz 6 (Sects. 8.1, 8.2) Duration: 20mn

Name: ID number:

Solve the homogeneous linear system

$$X' = \begin{pmatrix} 0 & 1 & -4/3 \\ -5/3 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} X$$

$X' = AX$

$\det(A - \lambda I) = 0 \iff$

$$\begin{vmatrix} -\lambda & 1 & -\frac{4}{3} \\ -\frac{5}{3} & -2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \iff$$

$$-\lambda(-2-\lambda)(1-\lambda) + \frac{5}{3}(1-\lambda) + \left(-\frac{4}{3}(2+\lambda)\right) = 0$$

$$-\lambda^3 - \lambda^2 - \lambda - 1 = 0$$

$$-(\lambda^2+1)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -1, \quad \lambda = \pm i$$

Case 1: $\lambda = -1$

$$\left(\begin{array}{ccc|c} 1 & 1 & -\frac{4}{3} & 0 \\ -\frac{5}{3} & -1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & \frac{10}{3} & 0 \\ 0 & -1 & \frac{10}{3} & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad K_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$$

$$\begin{cases} x + 2z = 0 \\ y - \frac{10}{3} z = 0 \end{cases} \Rightarrow \begin{cases} x = -2z \\ y = \frac{10}{3} z \end{cases}$$

$$K_1 \begin{pmatrix} -6 \\ 10 \\ 3 \end{pmatrix} \Rightarrow X_1 = \begin{pmatrix} -6 \\ 10 \\ 3 \end{pmatrix} e^{-t}$$

Case 2: $\lambda = i$

$$\left(\begin{array}{ccc|c} -i & 1 & -\frac{4}{3} & 0 \\ -\frac{5}{3} & -2-i & 0 & 0 \\ 1 & 0 & 1-i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1-i & 0 \\ 0 & 1 & -\frac{1}{3}+i & 0 \\ 0 & -2-i & \frac{5}{3}(1-i) & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1-i & 0 \\ 0 & 1 & -\frac{1}{3}+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad K_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$$

$$\begin{cases} x + z(1-i) = 0 \\ y + z\left(-\frac{1}{3}+i\right) = 0 \end{cases} \Rightarrow \begin{cases} x = -z(1-i) \\ y = -z\left(-\frac{1}{3}+i\right) \end{cases}$$

$$K_2 \begin{pmatrix} -1+i \\ \frac{1}{3}-i \\ 1 \end{pmatrix}$$

$$B_1 = \operatorname{Re} K_2 = \begin{pmatrix} -1 \\ 1/3 \\ 1 \end{pmatrix}, \quad B_2 = \operatorname{Im} K_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$X_2 = B_1 \cos t - B_2 \sin t = \begin{pmatrix} -1 \\ 1/3 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \sin t$$

$$X_3 = B_2 \cos t + B_1 \sin t = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} -1 \\ 1/3 \\ 1 \end{pmatrix} \sin t.$$

The general solution of the DE is

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3.$$