1. Write clearly.

2. Show all your steps.

3. No credit will be given to wrong steps.

4. Do not do messy work.

5. Calculators and mobile phones are NOT allowed in this exam.

6. Turn off your mobile.
1. Let $S$ be a subset of the vector space $V = \mathbb{R}^{n \times n}$ where $S = \{A : \text{where } A \text{ is } n \times n \text{ matrix such that } \det(A) = 0\}$. Is $S$ a subspace of $V$? Explain?
2. Does the matrix \[
\begin{bmatrix}
9 & 4 \\
1 & -2
\end{bmatrix}
\] belong to the spanning of \[
\begin{bmatrix}
3 & 4 \\
1 & 2
\end{bmatrix},
\begin{bmatrix}
0 & 2 \\
0 & 4
\end{bmatrix},
\begin{bmatrix}
0 & 2 \\
6 & 1
\end{bmatrix}
\]
3. Show that the functions $x, xe^x, x^2e^x$ form a linearly independent subset of the vector space $C[-\infty, \infty]$. 
4. Show that the set of vectors $X_1 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \ X_2 = \begin{bmatrix} -5 \\ 2 \\ -3 \end{bmatrix}, \ X_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ form a basis of $\mathbb{R}^3$, and then express the vectors $e_1, e_2, e_3$ of the standard basis in terms of these.

5. Let $A$ and $B$ be $m \times n$ and $n \times p$ matrices respectively. Prove that the row space of $AB$ is contained in the row spaces of $B$, and the column space of $AB$ is contained in the column space of $A$. What can one conclude about the ranks of $AB$ and $BA$?
6. Consider two ordered bases $\mathcal{B} = [v_1, v_2] = [(-4, 3), (2, -1)]$ and $\mathcal{C} = [u_1, u_2] = [(2, 1), (-4, 1)]$ of the vector space $\mathbb{R}^2$

(a) Find the transition matrix $S$ that describes the change of $\mathcal{B}$ to $\mathcal{C}$.
(b) Use part (a) to compute the coordinate vector $[x]_C$ where $x = 3v_1 + 2v_2$. 
7. Let \( A = \begin{bmatrix} 2 & 1 & 1 & 3 & 2 \\ -1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \)

(a) Find a basis for the row space of \( A \). What is the dimension of the row space.
(b) Find a basis for the column space of \( A \). What is the rank of \( A \)?
(c) Find a basis for the null space of \( A \). What is the dimension of \( N(A) \)?
8. Let $L_1 : U \to V$ and $L_2 : V \to W$ be linear transformations, and let $L = L_1 \circ L_2$ be the mapping defined by

\[ L(u) = L_2(L_1(u)), \quad \text{for each } u \in U. \]

Show that $L$ is a linear transformation mapping $U$ into $W$. 
9. A linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ is defined by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 - x_3 \\ -x_1 + x_3 \end{bmatrix}$$

Let $B$ and $C$ be the bases

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Find the matrix that represents $T$ with respect to these bases.
10. Find the kernel and the range of the linear operator $L$ on $P_3$, where $L(p(x)) = p(0)x + p(1)$