1. Do not do messy work.

2. Calculators and mobile phones are NOT allowed in this exam.

3. Turn off your mobile.
1. Let $S$ denote the set of all solutions of $y = y(x)$ of the homogeneous linear differential equation

$$y'' + 5y' + 6y = 0$$

defined in some interval $[a, b]$. Show that $S$ is a subspace of $C[a, b]$.

2. (5.1) Let $x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $y = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

(a) Determine the angle between $x$ and $y$?
(b) Determine the distance between $x$ and $y$. 
3. Find the equation of the plane that passes through the points $P_1 = (2, 3, 1)$, $P_2 = (5, 4, 3)$, and $P_3 = (3, 4, 4)$.
4. 5.2 Let

\[
A = \begin{bmatrix}
1 & 2 & 0 & 2 & 1 \\
3 & 6 & 1 & 9 & 6 \\
2 & 4 & 1 & 7 & 5 \\
\end{bmatrix}
\]

(a) Find basis and the dimension for \(N(A), R(A^T), N(A^T)\) and \(R(A)\).
(b) Find basis for \(N(A)^\perp, R(A^T)^\perp, N(A^T)^\perp\) and \(R(A)^\perp\).
5. 5.4 Given

\[ A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix} \]

determine the value of each of the following

(a) \( \langle A, B \rangle \)
(b) \( \|A\|_F \)
(c) \( \|A + B\|_F \)
6. Let \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 \), define

\[ \langle x, y \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2. \]

Prove it is an inner product space.

7. Show that

\[ \left| \int_a^b f(x)g(x)dx \right| \leq \left[ \int_a^b f^2(x)dx \right]^{1/2} \left[ \int_a^b g^2(x)dx \right]^{1/2} \]
8. In the inner product space $P_3(\mathbb{R})$ with

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$ 

Find the orthogonal complement of the subspace $S$ generated by 1 and $x$. 
9. Given the matrix \( A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \)

(a) Diagonalize \( A \).

(b) Use the result obtained in (a) to find \( A^{99} \).

10. If \( A \) is an invertible matrix with eigenvalues \( \lambda_1, \cdots, \lambda_n \). Show that the eigenvalues of \( A^{-1} \) are \( \lambda_1^{-1}, \cdots, \lambda_n^{-1} \).
11. Let \( A = \begin{bmatrix}
1 & 1 & 2 \\
1 & 2 & 3 \\
1 & 2 & 1 \\
1 & 1 & 6
\end{bmatrix} \)

(a) Find orthonormal basis for the column space \( S \) of the matrix \( A \).

(b) Find the QR-factorization of \( A \) by using part (a).
12. Let $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and let $L$ be a linear operator on $\mathbb{R}^2$ whose matrix representation with respect to the ordered basis \{u_1, u_2\} is $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(a) Determine the transition matrix from the basis \{v_1, v_2\} to the basis \{u_1, u_2\}.
(b) Find the matrix representation of $L$ with respect to \{v_1, v_2\}.
13. Given the quadratic equation

\[ x^2 + 4xy + y^2 + 3xy + y - 1 = 0 \]

find a change of coordinates so that the resulting equation represents a conic in standard position.
14. The function \( f(x, y) = (x^2 - 2x) \cos y \) has a critical point at \((1, \pi)\). Determine whether the given stationary point is local maximum, minimum or saddle point.
15. Find bases for the kernel and image of the linear transformation

\[ L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ where } L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 4x + 6y \end{pmatrix} \]