Time: 08:00 to 09:30 am

Name : ........................................ ......... .........

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Exercise 1.

(i) Find a basis and the dimension of the subspace

\[ S = \{(x, y, z, t) \in \mathbb{R}^4 \mid 2x - y = 0 \text{ and } 3z - t = 0\} \]

of \( \mathbb{R}^4 \).
(ii) Let $E = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = xy\}$. Determine whether $E$ is a subspace of $\mathbb{R}^2$. 
Exercise 2. Consider the nonhomogeneous system given by:

\[
\begin{align*}
&x_1 + 2x_2 + 3x_3 = 1 \\
&4x_1 + 5x_2 + 6x_3 = 1 \\
&7x_1 + 8x_2 + 9x_3 = 1
\end{align*}
\]

(a) Write the matrix form of the system.

(b) Find the reduced echelon form of the augmented matrix of the system.
(c) Solve the system. Does it have a unique solution?
Exercise 3. Let $a$ be a real number. Consider the system $AX = B$, where

$$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ 1 & 4 & 6 & 8 \\ 0 & 1 & 0 & 0 \\ 2 & 5 & 6 & 8 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 6 \\ 5 \\ a \end{pmatrix}$$

(i) Find $\text{rank}(A)$ and $\text{rank}([A:B])$. 
(ii) Find the value of $a$ such that the system $AX = B$ is consistent.
Exercise 4. Let \( A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 1 & 2 & -1 \end{pmatrix} \).

(i) Use Gauss-Jordan Elimination to find \( A^{-1} \).
(ii) Use $A^{-1}$ obtained in (i) to solve the following system:

\[
\begin{align*}
2x_1 - x_2 + x_3 &= 4 \\
3x_1 - x_3 &= 8 \\
x_1 + 2x_2 - x_3 &= -16
\end{align*}
\]
Exercise 5. Let \( A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \).

(i) Explain briefly why \( A \) is diagonalizable.

(ii) Show that
\[
\det(\lambda I_3 - A) = (\lambda + 6)(\lambda + 2)(\lambda - 3).
\]
(iii) Find an orthogonal matrix $P$ that diagonalizes $A$ and find the matrix $P^{-1}AP$. 