King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH311 - Advanced Calculus I
Exam II – Term 131 (2013–2014)

Exercise 1 (6 points)
Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to c}(f(x))^2 = L$.

1. Show that if $L = 0$, then $\lim_{x\to c} f(x) = 0$.
2. Show by example that if $L \neq 0$, then f may not have a limit at c.

Exercise 2 (6 points)
Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for x rational, and $g(x) = x + 3$ for x irrational. Find all points at which g is continuous.

Exercise 3 (5 points)
Give an example of each of the following:

1. a continuous $f : \mathbb{R} \to \mathbb{R}$ that is bounded but does not attain its bounds,
2. a continuous $f : (0, 1) \to \mathbb{R}$ that is unbounded,
3. a continuous $f : (0, 1) \to \mathbb{R}$ that is bounded but does not attain its bounds,
4. a function $f : [0, 1] \to \mathbb{R}$ that is bounded but does not attain its bounds,
5. a function $f : [0, 1] \to \mathbb{R}$ that is unbounded.

Exercise 4 (6 points)
Let $f : [a, b] \to \mathbb{R}$ be continuous and such that $f(x) \neq 0$ for all $x \in [a, b]$. Show that there exists a number $m > 0$ such that $|f(x)| \geq m$ for all $x \in [a, b]$.

Exercise 5 (6 points)
Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f is differentiable for all $x \in \mathbb{R}$, but that f' is not continuous at 0.[Use the definition of f' at 0, use the standard formula for $x \neq 0$.]

Exercise 6 (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $|f'(x)| \leq M$ for all x, for some number M. Show that f is uniformly continuous on $\mathbb{R}$. Deduce that $f(x) = \dfrac{x}{(1 + x^2)}$ is uniformly continuous on $\mathbb{R}$.

Exercise 7 (7 points)
Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$.

1. For a fixed $h > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = h^2[f(h) - f(x) - (h - x)f'(x)] - (h - x)^2[f(h) - f(0) - hf'(0)]$$

   By applying Rolle's theorem to g on $[0, h]$, show that there is a number $c \in (0, h)$ such that

$$f(h) = f(0) + f'(0)h + \frac{1}{2}f''(c)h^2. \qquad (1)$$

2. Suppose that there are positive numbers M_0 and M_2 such that $|f(x)| \leq M_0$ and $|f''(x)| \leq M_2$ for all $x \in \mathbb{R}$. Prove that $|f'(0)| \leq 2\sqrt{M_0 M_2}$.

   [Hint: Use (1) to estimate $|f'(0)|$ and choose a suitable value of h.]

Exercise 8 (`8 points`)
Fix an integer $n \geq 1$. Let f_n be the function defined by

$$f_n(x) = x^n + x^{n-1} + \ldots + x - 1$$

1. Prove that the equation

$$f_n(x) = x^n + x^{n-1} + \ldots + x - 1 = 0$$

   has a *unique* positive solution. Let a_n denote this solution. Show that $a_n \in (0, 1]$.

2. Compute a_1 and a_2.

3. Prove that $f_n(a_{n+1}) < 0$, deduce that the sequence $(a_n)_{n \geq 1}$ is decreasing.

4. Show that

$$a_n^{n+1} - 2a_n + 1 = 0$$

   and deduce that $\lim_{n \to \infty} a_n = \frac{1}{2}$.