King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 513  Final Exam
The First Semester of 2013-2014 (131)

Time Allowed: 180 Minutes

Name: ___________________________  ID#: ______________
Instructor: ______________________  Sec #: _______  Serial #: _______

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

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Q:1 (14 points) Find eigenvalues and eigenfunctions of the boundary value problem
\[ x^2y'' + xy' + \lambda y = 0, \text{ with } y(1) = 0, y(4) = 0, \text{ for } \lambda = \alpha^2. \] Also write the orthogonality relation of the eigenfunctions.
Q:2 (14 points) Use Laplace transformation method to solve the wave equation

\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = xe^{-t}, \quad 0 < x < \infty, \quad t > 0, \]

with the initial conditions \( u(x, 0) = 1, \ u_t(x, 0) = 0, \)

and the boundary conditions \( u(0, t) = \cos t, \ \lim_{x \to \infty} |u(x, t)| \sim x^n, \)

for some finite \( n, \ t > 0. \)
Q:3 (14 points) Solve the heat equation
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \]
subject to the following initial and boundary conditions
\[ u(x,0) = 4 \text{ for } 0 < x < \pi \quad \text{and} \quad u(0,t) = 2, \quad u_x(\pi,t) = 0 \text{ for } t > 0. \]
Q:4 (15 points) Show that the steady-state temperature in the circular cylinder of radius 2 and height 4 is \( u(r, z) = 2z \) by solving

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}, \quad 0 < r < 2, \quad 0 < z < 4
\]

with initial and boundary conditions

\( u_r(2, z) = 0, \quad 0 < z < 4 \) and \( u(r, 0) = 0, \quad u(r, 4) = 8 \). (Hint: \( J_0'(2) = 0 \Rightarrow J_1(2) = 0 \))
Q:5 (15 points) Find the steady-state temperature in a hemisphere of radius 2 by solving

$$\frac{\partial^2 u}{\partial r^2} + 2 \frac{\partial u}{r \partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \quad 0 < r < 2, \quad 0 < \theta < \frac{\pi}{2}$$

with initial and boundary conditions

$$u(r, \frac{\pi}{2}) = 0, \quad 0 < r < 2 \quad \text{and} \quad u(2, \theta) = 2 + \cos(\theta), \quad 0 < \theta < \frac{\pi}{2}.$$ 

(Hint: $P_n(0) = 0$ only for odd values of $n$)
Q:6 (14 points) Use appropriate Fourier transform to solve
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0, \]
with initial and boundary conditions
\[ u_y(x, 0) = 0, \quad 0 < x < \pi, \text{ and } \quad u(0, y) = 0, \quad u(\pi, y) = e^{-y}, \quad y > 0. \]
Q:7 (14 points) Solve the nonhomogeneous linear system using variation of parameters method

\[ X' = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} X + \begin{bmatrix} \tan t \\ 1 \end{bmatrix} \]