Math 550 – Linear Algebra (Term 131)

Final Exam

Notes:
- Math Students: Solve 6 problems (out of 7).
- Non-Math Students: Solve 5 problems (out of 7).
- Duration = 3 hours.
- Each problem is worth 20 points.

(1) Let $T$ be a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix
\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{bmatrix}.
\]
(a) Prove that $T$ has no cyclic vector.
(b) What is the $T$-cyclic subspace generated by the vector $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

(2) Let $n$ be an integer $\geq 2$ and let $N$ be an $n \times n$ matrix over the field $F$ such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that $N$ has no square root (i.e., there is no $n \times n$ matrix $A$ such that $A^2 = N$).

(3) Let $A$ be an $n \times n$ matrix with real entries such that $A^2 + I = 0$.
(a) Prove $n$ is even.
Let $n = 6$.
(b) Give the rational form of $A$.
(c) Describe explicitly the cyclic decomposition.

(4) Give all possible Jordan forms for linear operator $T$ with minimal polynomial $x^2(x - 1)^2$ and characteristic polynomial $x^4(x - 1)^4$. 
(5) Let $V$ be the vector space of the polynomials over $\mathbb{R}$ of degree $\leq 3$, with the inner product $(f | g) = \int_0^1 f(x) g(x) dx$. 

Let $t$ be a real number, find explicitly $g_t \in V$ such that $(f | g_t) = f(t)$ for all $f$ in $V$. (Hint: Use Theorem 6 of Section 8.3)

(6) Let $V$ be a finite-dimensional inner product space. For each $\alpha, \beta$ in $V$, let $T_{\alpha, \beta}$ be the linear operator on $V$ defined by $T_{\alpha, \beta}(\gamma) = (\gamma | \beta)\alpha$. Show that

(a) $T_{\alpha, \beta}^* = T_{\beta, \alpha}$
(b) Trace($T_{\alpha, \beta}$) = $(\alpha | \beta)$
(c) $T_{\alpha, \beta} T_{\gamma, \delta} = T_{\alpha, (\beta | \gamma)\delta}$
(d) Under what conditions is $T_{\alpha, \beta}$ self-adjoint?

(7) Let $f$ and $g$ be bilinear forms on a finite dimensional vector space $V$. Suppose that $g$ is non-singular.

(a) Show that there exist unique linear operators $T_1$ and $T_2$ on $V$ such that:
   
   $f(\alpha, \beta) = g(T_1 \alpha, \beta) = g(\alpha, T_2 \beta)$ for all $\alpha, \beta$.

(Hint: For a bilinear form $\varphi$ use: $\varphi(\alpha, \beta) = X^TAY$ where $[\varphi] = A, [\alpha] = X, [\beta] = Y$)

(b) Explain why (a) is not true if $g$ is singular.

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