King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 101 - Exam I - Term 132

Duration: 120 minutes

Name: Key

ID Number:

Section Number:

Serial Number:

Class Time:

Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 9 Problems)

Page Number	Points	Maximum Points
1		12
2		14
3		12
4		12
5		20
6		10
7		10
8		10
Total		100

1. Find the limit if it exists. Justify your answer

a) (4 points) $\lim_{x \to -4} \left(\frac{2x+8}{x^2+x-12} \right)$

$$\lim_{x \to -4} \frac{2x+8}{x^2+x-12} = \lim_{x \to -4} \frac{2(x+4)}{(x+4)(x-3)} \quad (2 \text{ pts})$$

$$= \lim_{x \to -4} \frac{2}{(x-3)} \quad (1 \text{ pt})$$

$$= -\frac{2}{7} \quad (1 \text{ pt})$$

b) (4 points) $\lim_{x \to 1^+} \left(\frac{|x^2-3x+2|}{x^2-1} \right)$

$$\lim_{x \to 1^+} \frac{|x^2-3x+2|}{x^2-1} = \lim_{x \to 1^+} \frac{|(x-1)(x-2)|}{(x-1)(x+1)} \begin{matrix} \leftarrow (1 \text{ pt}) \\ \leftarrow (1 \text{ pt}) \end{matrix}$$

$$= \lim_{x \to 1^+} \frac{|x-1|\,|x-2|}{(x-1)(x+1)}$$

$$= \lim_{x \to 1^+} \frac{\cancel{(x-1)}\,|x-2|}{\cancel{(x-1)}(x+1)} = \frac{1}{2} \quad (1 \text{ pt})$$

(1 pt)

c) (4 points) $\lim_{x \to 0} \left(x^2 e^{\sin\left(\frac{\pi}{x}\right)} \right)$

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$$\Rightarrow e^{-1} \leq e^{\sin\left(\frac{\pi}{x}\right)} \leq e^{1} \leftarrow (1 \text{ pt})$$

$$\Rightarrow \frac{x^2}{e} \leq x^2 e^{\sin\left(\frac{\pi}{x}\right)} \leq e x^2 \leftarrow (1 \text{ pt})$$

Since $\lim_{x \to 0} \frac{x^2}{e} = 0 = \lim_{x \to 0} e x^2$, by squeeze theorem $\lim_{x \to 0} \left(x^2 e^{\sin\left(\frac{\pi}{x}\right)} \right) = 0$ (1 pt)

$\uparrow$ (1 pt)

d) (4 points) $\lim_{x\to\infty}(3x-1)(\sqrt{x^2+1}-x)$

$$\lim_{x\to\infty} (3x-1)\left(\sqrt{x^2+1}-x\right)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \quad (1\text{ pts})$$

$$=\lim_{x\to\infty} (3x-1)\cdot\frac{1}{\sqrt{x^2+1}+x} \quad (1\text{ pt})$$

$$=\lim_{x\to\infty}\frac{3x-1}{x\sqrt{1+\frac{1}{x^2}}+x} = \lim_{x\to\infty}\frac{3-\cancel{\frac{1}{x}}^{\to 0}}{\sqrt{1+\cancel{\frac{1}{x^2}}^{\to 0}}+1} = \frac{3}{2} \quad (1\text{ pt})$$

(1 pt)

e) (4 points) $\lim_{h\to 0}\left(\frac{\sqrt{4-\sin 3h}-2}{h}\right)$

$$\lim_{h\to 0}\frac{\sqrt{4-\sin 3h}-2}{h}\cdot\frac{\sqrt{4-\sin 3h}+2}{\sqrt{4-\sin 3h}+2} \quad (1\text{ pt})$$

$$=\lim_{h\to 0}\frac{-\sin 3h}{h\left(\sqrt{4-\sin 3h}+2\right)} = 3\lim_{h\to 0}\frac{-\sin 3h}{3h}\cdot\frac{1}{\sqrt{4-\sin 3h}+2} \quad (1\text{ pt})$$

$$=(-3)\left(\frac{1}{4}\right) = -\frac{3}{4} \quad (1\text{ pt})$$

(1 pt)

2. (6 points) If $\lim_{x\to 2}\left(\frac{f(x)-2x+3}{x-2}\right)=1$, find $\lim_{x\to 2} f(x)$

For the $\lim_{x\to 2}\frac{f(x)-2x+3}{x-2}$ to exist, we must have $\lim_{x\to 2}\left(f(x)-2x+3\right)=0$. Othewise, the Limit DNE. (3 pts)

$$\Rightarrow \left(\lim_{x\to 2} f(x)\right)-4+3=0 \quad (2\text{ pts})$$

$$\Rightarrow \lim_{x\to 2} f(x) = 1 \quad (1\text{ pt})$$

Remark: Other solutions are possible.

3. (12 points)Let

$$f(x) = \begin{cases} [|x|] - 3 & , \quad -2 \le x < -1 \\ x + 1 & , \quad -1 \le x < 2 \\ 5x - 7 & , \quad x > 2, \end{cases}$$

where $[|y|]$ is the greatest integer less than or equal to y. Is f continuous at $x = -1$ and $x = 2$. If not, what kind of discontinuity does it have at these points.

Justify your answer using limits.

$x = -1$:

$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} \left([\![x]\!] - 3 \right) = -5$ (2 pts)

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} (x + 1) = 0$ (1 pt)

Since $\lim\limits_{x \to -1^+} f(x) \neq \lim\limits_{x \to -1^-} f(x) \Rightarrow \lim\limits_{x \to -1} f(x)$ DNE. (1 pt.)

(1 pt)

So, f is not continuous at $x = -1$. It has Jump discontinuity at $x = -1$. (1 pt)

$x = 2$:

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (5x - 7) = 3$, (1 pt)

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x + 1) = 3$ (1 pt)

$\Rightarrow \lim\limits_{x \to 2} f(x) = 3$.

Since $2 \notin$ domain of f and the $\lim\limits_{x \to 2} f(x)$ exists (2 pts) $\Rightarrow f$ has a removable discontinuity at $x = 2$. (2 pts)

4. (12 points)Find all the asymptotes of the graph of the function $f(x) = \dfrac{x^3 - 5x^2 + 6x}{x^2 - 4}$. Justify your answer using limits.

$$f(x) = \frac{x(x-2)(x-3)}{(x-2)(x+2)}, \quad x \neq -2, 2.$$

$$\Rightarrow f(x) = \frac{x(x-3)}{x+2}, \quad x \neq -2, 2. \quad \text{(2pts)}$$

H.A: $\lim\limits_{x \to \infty} \dfrac{x(x-3)}{x+2} = \infty$, (1pt)

$\lim\limits_{x \to -\infty} \dfrac{x(x-3)}{x+2} = -\infty$ (1pt)

$\Rightarrow$ f has No horizontal asymptotes. (1pt)

V.A: $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x(x-3)}{x+2} = -\dfrac{1}{2}$ (1pt)

So, $x = 2$ is not a vertical asymptote. (1pt)

$\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} \dfrac{x(x-3)}{x+2} = -\infty$ (1 pts)

$\therefore$ $x = -2$ is a vertical asymptote (1pt)

Oblique (slant) asymptote

$$\begin{array}{r|l} & x - 5 \\ \hline x+2 & x^2 - 3x \\ & x^2 + 2x \\ \hline & -5x \\ & -5x - 10 \\ \hline & 10 \end{array}$$

$\therefore$ $f(x) = (x-5) + \dfrac{10}{x+2}$ (1pt)

So, $y = x - 5$ is a slant asymptotes. (1pt)

5. (10 points)Use the Intermediate Value Theorem to show that the graphs of the functions
$f(x) = \ln x$ and $g(x) = e^{-x}$ intersect at a point whose x- coordinate lies in the interval $[1, e]$.

Let $h(x) = f(x) - g(x) = \ln x - e^{-x}$. (2 pts)

(2 pts) $\left\{\right.$ h is continuous on $[1, e]$ since f and g are continuous there. Also,

$h(1) = \ln 1 - e^{-1} = -\frac{1}{e} < 0$, (1 pt)

$h(e) = \ln e - e^{-e} = 1 - \frac{1}{e^e} > 0$, (1 pt)

(1 pt) $\rightarrow$ Since $h(1) < 0 < h(e)$ and h is cont. on $[1, e]$, then

(3 pts) $\rightarrow$ by Intermediate Value Theorem there is a number $c \in (1, e)$
$\rightarrow$ such that $h(c) = 0 \Rightarrow f(c) = g(c)$.

6. (10 points)Use $\varepsilon\delta$-definition of limits to prove that $\lim_{x \to -1} (2 - 3x) = 5$.

Let $\epsilon > 0$ be given. We need to find a number $\delta > 0$ Such that

$|2 - 3x - 5| < \epsilon$ whenever $0 < |x+1| < \delta$

or $|-3 - 3x| < \epsilon$ " "

or $3|x+1| < \epsilon$ " "

or $|x+1| < \epsilon/3$ " ".

Take $0 < \delta \leq \epsilon/3$. Now,

If $|x+1| < \frac{\epsilon}{3}$, then

$|2 - 3x - 5| = |-3 - 3x| = 3|x+1| < 3 \cdot \frac{\epsilon}{3} = \epsilon$.

7. (10 points) Where is $f(x) = \dfrac{\sqrt{x^2 - 16}}{x^2 - 3x - 10}$ continuous?

$f(x)$ is continuous over its domain. (2 pts)

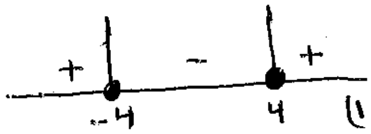
Domain of f :

$x^2 - 16 \geq 0$ and $x^2 - 3x - 10 \neq 0$ (2 + 2 pts)

$\Rightarrow (x-4)(x+4) \geq 0$ and $(x-5)(x+2) \neq 0$

$\Rightarrow (x-4)(x+4) \geq 0$ and $x \neq -2, 5$.
(1 pt)

$(x+4)(x-4)$



(1 pt)

$\therefore D_f = (-\infty, -4] \cup [4, 5) \cup (5, \infty)$ (2 pts)

So f is cont. on $(-\infty, -4]$, $[4, 5)$ and $(5, \infty)$.

8. (10 points)Let $f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x^3}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Find $f'(0)$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \quad (2\text{pts})$$

$$= \lim_{x \to 0} \frac{x^3 \sin\left(\frac{1}{x^3}\right) - 0}{x} \quad (3\text{pts})$$

$$= \lim_{x \to 0} x^2 \sin\left(\frac{1}{x^3}\right) \quad (1\text{ pt})$$

Now, since $-x^2 \leqslant x^2 \sin\left(\frac{1}{x^3}\right) \leqslant x^2$ (1pt)

and $\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$, then by (1pt)

Sequeeze theorem $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^3}\right) = 0$ (1pt)

$\therefore f'(0) = 0$. (1pt)

9. (10 points)Find an equation of the line that is tangent to the curve of $f(x) = x + \sqrt{x}$ at the point $(1, 2)$. (**You must use limits**).

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \quad (1\text{ pt})$$

$$= \lim_{h\to 0} \frac{(1+h) + \sqrt{1+h} - 2}{h} \quad (1\text{ pt})$$

$$= \lim_{h\to 0} \frac{\sqrt{1+h} + h - 1}{h} \cdot \frac{\sqrt{1+h} - (h-1)}{\sqrt{1+h} - (h-1)} \quad (1\text{ pt})$$

$$= \lim_{h\to 0} \frac{(1+h) - (h-1)^2}{h\left(\sqrt{1+h} - h + 1\right)} \quad (1\text{ pt})$$

$$= \lim_{h\to 0} \frac{1 + h - h^2 + 2h - 1}{h\left(\sqrt{1+h} - h + 1\right)} \quad (1\text{ pt})$$

$$= \lim_{h\to 0} \frac{-h^2 + 3h}{h\left(\sqrt{1+h} - h + 1\right)} \quad (1\text{ pt})$$

$$= \lim_{h\to 0} \frac{-\cancel{h}(h-3)}{\cancel{h}\left(\sqrt{1+h} - h + 1\right)} = \frac{3}{2}. \quad (2\text{ pts})$$

So, an equation of the tangent Line is

$$y - 2 = \frac{3}{2}(x - 1) \quad (2\text{ pts})$$

or

$$y = \frac{3}{2}x - \frac{3}{2} + 2 = \frac{3}{2}x + \frac{1}{2}$$

$$\therefore\ y = \frac{3}{2}x + \frac{1}{2}$$