King Fahd University Petroleum and Minerals
Department of Mathematics and Statistics

MASTER

MATH 201 - Term 132 - Exam II

MASTER

Duration: 120 minutes

Name:________________________ ID Number:________________________

Section Number:__________________ Serial Number:________________________

ClassTime: ________________________ Instructor's Name:____________________

General Instructions:

1. Calculators and Mobiles are not allowed.
2. This exam consists of two parts: Written and Multiple Choice.

Parts	Points	Maximum Points
Written		50
MCQ		50
Total		100

Part I: Written Questions

Instructions for Written Questions

1. This part has 5 written questions.
2. Answer the questions in the space provided.
3. Show your work or explain your solution. Points will be deducted for results without work.
4. Write clearly. Points may be deducted for poor presentation.
5. No credits will be given to wrong steps.

Question Number	Points	Maximum Points
1		11
2		9
3		10
4		10
5		10
Total		50

1. Let $f(x, y) = \sin^{-1}(y - 2x)$.

(a) (5-points) Find the domain of f and its boundary.

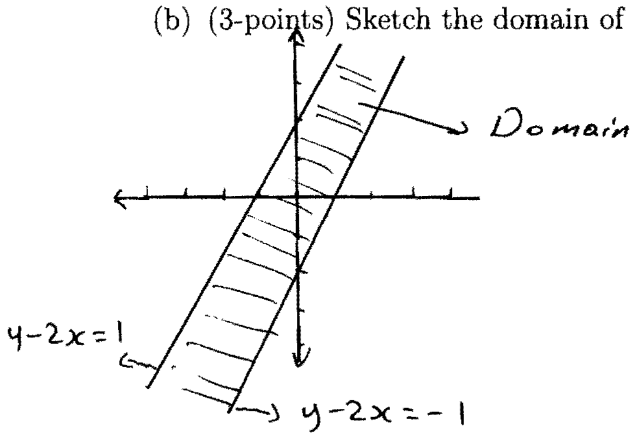
The domain of $\sin^{-1}$ is $[-1, 1]$ (1 pt)

Therefore the domain of $(x,y) \longmapsto \sin^{-1}(y-2x)$ is

$D_f = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq y - 2x \leq 1\}$ (2 pt).

The boundaries of f are $\underbrace{y-2x=-1}_{(1\ pt)}$ and $\underbrace{y-2x=1.}_{(1\ pt)}$

(b) (3-points) Sketch the domain of f.



each line (1 pt).

Shaded region (1 pt).

* if any of the lines are not included (-1 pt)

(c) (2-points) Is the domain of f bounded? Explain.

The domain of f is not bounded (1 pt), because it cannot be contained in a disk with fixed radius. (1 pt).

(d) (1-points) Find the range of f.

The range of f is $[-\pi/2, \pi/2]$. (1 pt).

2. (9-points) Find the parametric equations for the line in which the planes $x - 2y + 4z = 2$ and $x + y - 2z = 5$ intersect.

(1pt) the normal vector of the plane $x-2y+4z=2$ is $\vec{n}_1 = \langle 1,-2,4 \rangle$

(1pt) the normal vector of the plane $x+y-2z=5$ is $\vec{n}_2 = \langle 1,1,-2 \rangle$

(1pt) the direction vector of the line in the intersection is $\vec{n}_1 \times \vec{n}_2$

(1pt) $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = \langle 0, 2, 1 \rangle$

(2 pts) point at the intersection: Assume $z=0$. Then $\begin{cases} x-2y=2 \\ x+y=5 \end{cases}$

$\Rightarrow \begin{cases} x-2y=2 \\ 3y=3 \end{cases} \Rightarrow \{ y=1, x=4.$

Then a point at the intersection is $(4,1,0)$.

Eqn of the line: $\underbrace{x=4}_{(1pt)}$, $\underbrace{y=2t+1}_{(1pt)}$, $\underbrace{z=t}_{(1pt)}$.

3. (10-points) Find the distance from the line $x = 2 + t$, $y = 1 + t$, $z = 4 - t/3$ to the plane $-x + 2y + 3z = 11$.

(1 pt) direction vector of the line is $\vec{d} = \langle 1, 1, -1/3 \rangle$

(1 pt) normal vector of the plane is $\vec{n} = \langle -1, 2, 3 \rangle$

$\underbrace{\vec{d} \bullet \vec{n} = -1 + 2 - 1 = 0}_{(1\,pt)} \Rightarrow \underbrace{\text{the line is parallel to the plane}}_{(1\,pt)}$.

(1 pt) choose a point on the line : $P(2, 1, 4)$

(1 pt) choose a point on the plane : $S(1, 3, 2)$

(1 pt) $\vec{SP} = \langle 1, -2, 2 \rangle$.

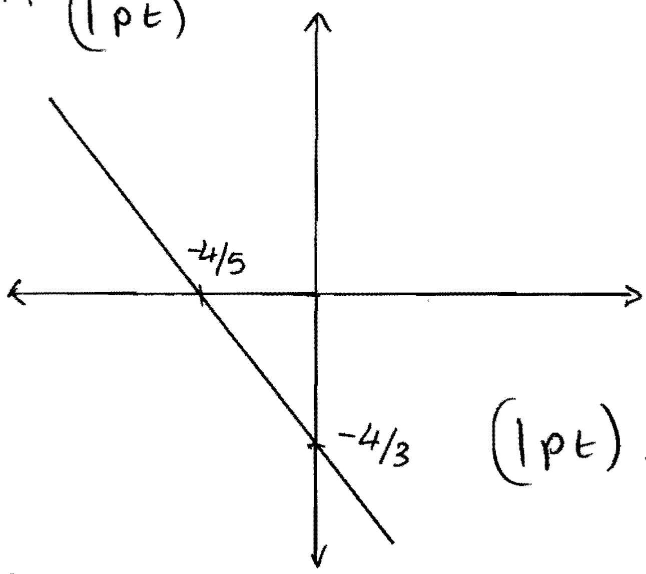
$$\underbrace{\text{Distance} = \left| \vec{SP} \bullet \frac{\vec{n}}{|\vec{n}|} \right|}_{(2\,pts)} = \frac{1}{\sqrt{1+4+9}} \left| \langle 1, -2, 2 \rangle \bullet \langle -1, 2, 3 \rangle \right|$$

$$= \frac{1}{\sqrt{14}} |-1 - 4 + 6| = \underbrace{\frac{1}{\sqrt{14}}}_{(1\,pt)}$$

4. a) (4-points) Find and sketch the level curve of $f(x, y) = \dfrac{y - x}{x + y + 1}$ that passes through the point $(-2, 2)$.

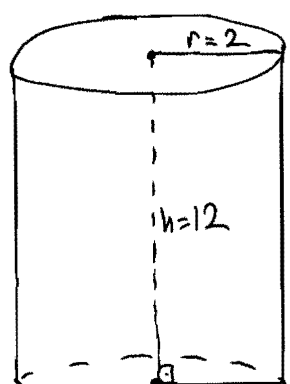
$$f(-2,2) = \frac{2-(-2)}{-2+2+1} = 4. \quad (1\text{ pt})$$

$$\frac{x-y}{x+y+1} = 4 \;(1\text{ pt}) \Rightarrow y - x = 4x + 4y + 4 \Rightarrow \underbrace{5x + 3y + 4 = 0}_{\text{eqn of the level curve}} \; (1\text{pt}).$$



(1pt).

b) (6-points) Consider a soda can in the shape of a right cylinder of radius 2cm and height 12cm. The can is coated with a layer of paint 0.1cm thick. Use differentials to estimate the volume of the paint.



Volume $= V(r,h) = \pi r^2 h$

$dV = 2\pi r h\, dr + \pi r^2 dh$ (1 pt)

Estimated volume of paint $= dV$. (1 pt).

$dr = 0.1$ (1 pt) $\quad dh = 0.2$. (1 pt) $\quad r = 2 \quad h = 12$.

$dV = 2\pi . 2 . 12 . (0.1) + \pi . 4 . (0.2)$ (1 pt).

$dV = 5.6\pi$ (1 pt).

5. (10-points) Find the parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $x^2 + 4y^2 + z^2 = 9$ at the point $(1, -1, 2)$.

* Let $f(x,y,z) = x^2 + y^2 - z$ and consider the paraboloid as level surface of f.

$\underbrace{\nabla f(x,y,z) = \langle 2x, 2y, -1\rangle}_{(1\text{ pt})} \Rightarrow \underbrace{\nabla f(1,-1,2) = \langle 2, -2, -1\rangle}_{(1\text{ pt})}.$

* Let $g(x,y,z) = x^2 + 4y^2 + z^2$ and consider the ellipsoid as level surface of g.

$\underbrace{\nabla g = \langle 2x, 8y, 2z\rangle}_{(1\text{ pt})} \Rightarrow \underbrace{\nabla g(1,-1,2) = \langle 2, -8, 4\rangle}_{(1\text{ pt})}.$

(1 pt) * The direction vector of the tangent line to the curve of intersection of the paraboloid and the ellipsoid is $\vec{d} = \vec{\nabla} f(1,-1,2) \times \vec{\nabla} g(1,-1,2)$

$$\vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ 2 & -8 & 4 \end{vmatrix} = \langle -16, -10, -12\rangle \quad (1\text{ pt}).$$

* Eqn of the line: using the point $\overbrace{(1,-1,2)}^{(1\text{ pt})}$ is

$\underbrace{x = 1 - 16t}_{(1\text{ pt})}, \quad \underbrace{y = -1 - 10t}_{(1\text{ pt})}, \quad \underbrace{z = 2 - 12t}_{(1\text{ pt})}.$

Part II: Multiple Choice Questions

Instructions for Multiple Choice Questions

1. This part has 10 multiple choice questions.
2. Each question carries 5 points.
3. No partial credit.
4. Transfer all your answers to the table below.

Question 1	A	B	C	D	E
Question 2	A	B	C	D	E
Question 3	A	B	C	D	E
Question 4	A	B	C	D	E
Question 5	A	B	C	D	E
Question 6	A	B	C	D	E
Question 7	A	B	C	D	E
Question 8	A	B	C	D	E
Question 9	A	B	C	D	E
Question 10	A	B	C	D	E

1. An equation of the plane containing the line $x = 5+t$, $y = 1+3t$, $z = 4t$ and perpendicular to the plane $-5x + y + z = 5$ is

(A) $x + 21y - 16z = 26$

(B) $-5x + 21y + 16z + 4 = 0$

(C) $x + 10y - 5z = 15$

(D) $5x - y + 8z = 22$

(E) $x - 10y + z + 5 = 0$

direction vector of the line $\vec{d} = \langle 1, 3, 4 \rangle$.

normal vector of the plane $\vec{n} = \langle -5, 1, 1 \rangle$

Let $\vec{u}$ be the normal vector of the plane P we want to find.

Since the line is in P $\vec{d} \perp \vec{u}$

Since the plane is $\perp P$ $\vec{n} \perp \vec{u}$.

So we can take $\vec{u} = \vec{d} \times \vec{n} = \langle -1, -21, 16 \rangle$

Choose a point on the line : $(5, 1, 0)$.

The equation of P is:

$-1(x-5) - 21(y-1) + 16z = 0$.

$x + 21y - 16z = 26$

2. For the surface given by the equation $\underbrace{x^2 + 4y^2 - 4z^2 = 0}_{\text{elliptical cone}}$, decide which one of the following statements is **FALSE**:

(A) The cross-section of the surface in the plane $y = 0$ is a hyperbola. X $\Rightarrow x^2 = 4z^2 \rightarrow$ cross sections are two lines $x = \pm 2z$

(B) The cross-section of the surface in the plane $z = 1$ is an ellipse. ✓ $\Rightarrow x^2 + 4y^2 = 4 \Rightarrow$ ellipse

(C) The surface intersects the xy-plane only at the origin. $\Rightarrow x^2 + 4y^2 = 0 \Rightarrow x = y = 0$

(D) The cross-section of the surface in the plane $x = 2$ is a hyperbola. $\Rightarrow 4 + 4y^2 - 4z^2 = 0$

$z^2 - y^2 = 1 \Rightarrow$ hyperbola

(E) The surface is an elliptical cone. ✓

3. If L is the limit of $f(x,y) = \tan^{-1}\left(\dfrac{|x|+|y|}{x^2+y^2}\right)$ as (x,y) approaches to $(0,0)$, then

⇒ (A) $L = \pi/2$

(B) $L = \pi/4$

(C) $L = 0$

(D) $L = 1$

(E) L does not exist

Using polar coordinates

$$\frac{|x|+|y|}{x^2+y^2} = \frac{|r|(|\sin\theta|+|\cos\theta|)}{r^2} = \frac{|\sin\theta|+|\cos\theta|}{|r|}$$

Since $|\sin\theta|+|\cos\theta| \geq 1$ and $\frac{1}{|r|} \xrightarrow[r\to 0]{} \infty$,

$$\lim_{r\to 0} \frac{|\sin\theta|+|\cos\theta|}{|r|} = \infty.$$

Then $\lim_{r\to 0} \tan^{-1}\left(\frac{|\sin\theta|+|\cos\theta|}{|r|}\right) = \pi/2$.

4. The distance from the point of intersection of the two lines
$L1: x = t-1,\ y = t+2,\ z = 1-t$ and $L2: x = 1-4s,\ y = 1+2s,\ z = 2-2s$
to the xy-plane is

⇒ (A) 1

(B) $\frac{1}{2}$

(C) 2

(D) $\frac{3}{2}$

(E) -1

intersection point

$$\left.\begin{array}{l} t-1 = 1-4s \\ t+2 = 1+2s \\ 1-t = 2-2s \end{array}\right\} \Rightarrow \left.\begin{array}{l} t+4s = 2 \\ t-2s = -1 \end{array}\right\} \Rightarrow 6s = 3 \Rightarrow s = 1/2 \quad t = 0.$$

We verify that these values satisfy the last eqn.

$1-0 = 2-2.\frac{1}{2}$ ✓

Then these lines intersect when $t=0$ or $s=1/2$ at the point $(-1,2,1)$.
The distance of this point to the xy-plane is its z-coordinate. So the distance is 1.

5. If $f(x,y) = x^5 + x^3y^2 - xe^{y/x}$, then $f_{xy}(-1,1)$ is equal to

⟹ (A) $6 + \frac{1}{e}$

(B) $8 - \frac{1}{e}$

(C) $9 + \frac{2}{e}$

(D) $5 - \frac{2}{e}$

(E) $1 - \frac{1}{e}$

Notice that calculating the partial derivative with respect to y first is easier.

$$f_y(x,y) = 2x^3y - e^{y/x}$$

$$f_{xy}(x,y) = 6x^2y + \frac{y}{x^2}e^{y/x}$$

$$f_{xy}(-1,1) = 6 + e^{-1} = 6 + \frac{1}{e}.$$

6. If $w = \dfrac{x^2 - y^3}{z^5}$ where x, y, z, and w are differentiable functions with $x = u^2 + v^2$, $y = \dfrac{v}{u}$, and $z = u^v$, then $\left.\dfrac{\partial w}{\partial u}\right]_{(u,v)=(1,2)}$ is equal to

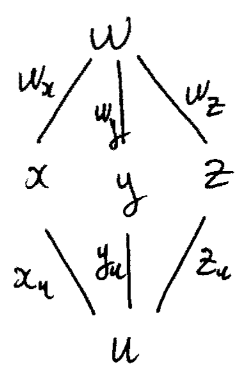
⟹ (A) -126

(B) -160

(C) -41

(D) 129

(E) 214



$$\frac{\partial w}{\partial u} = w_x x_u + w_y y_u + w_z z_u$$

$$= \frac{4xu}{z^5} + \frac{3vy^2}{u^2z^5} - \frac{5vu^{v-1}(x^2-y^3)}{z^6}$$

when $u=1$ and $v=2$, $x=5$, $y=2$, and $z=1$.

$$\left.\frac{\partial w}{\partial u}\right]_{(u,v)=(1,2)} = 20 + 24 - 5(17).2 = 44 - 170 = -126.$$

7. A direction of zero change in $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ at $(2,-1)$ is

$\Rightarrow$ (A) $\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$

(B) $\langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \rangle$

(C) $\langle \frac{3}{5}, \frac{4}{5} \rangle$

(D) $\langle \frac{4}{5}, \frac{-3}{5} \rangle$

(E) $\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

We want to find a unit vector $\vec{u}$ so that $D_{\vec{u}} f(2,-1) = 0$.

$D_{\vec{u}} f(2,-1) = \vec{u} \cdot \langle f_x(2,-1), f_y(2,-1) \rangle$

$f_x(x,y) = \dfrac{6xy(x^2+y^2) - 6x^3y}{(x^2+y^2)^2} \Rightarrow f_x(2,-1) = -\dfrac{12}{25}$

$f_y(x,y) = \dfrac{3x^2(x^2+y^2) - 6x^2y^2}{(x^2+y^2)^2} \Rightarrow f_y(2,-1) = \dfrac{36}{25}$

Then $\langle f_x(2,-1), f_y(2,-1) \rangle = \langle -\frac{12}{25}, \frac{36}{25} \rangle = \frac{12}{25}\langle -1, 3 \rangle$

So we can take $\vec{u} = \langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$

8. If $2^{xz} + \tan^{-1}(y+z) = 2$, then $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ at $(1,-1,1)$ is equal to

$\Rightarrow$ (A) -1

(B) $\dfrac{1}{1+\ln 4}$

(C) $-\dfrac{\ln 4}{1+\ln 4}$

(D) $\dfrac{1}{3}$

(E) $-\dfrac{4}{3}$

Let $F(x,y,z) = 2^{xz} + \tan^{-1}(y+z)$.

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2^{xz}\ln 2 \cdot z}{2^{xz}\ln 2 \cdot x + \frac{1}{1+(y+z)^2}}$

$\left.\dfrac{\partial z}{\partial x}\right|_{(1,-1,1)} = -\dfrac{\ln 4}{\ln 4 + 1}$

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{\frac{1}{1+(y+z)^2}}{2^{xz}\ln 2 \cdot x + \frac{1}{1+(y+z)^2}}$

$\left.\dfrac{\partial z}{\partial y}\right|_{(1,-1,1)} = -\dfrac{1}{\ln 4 + 1}$

$\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = -1$.

9. If P is the tangent plane to the surface $xz = \ln(y + e^z)$ at the point $(1, 0, -1)$, then decide which one of the below statements is **FALSE**:

normal of P normal of xy-plane.

$\Rightarrow$ (A) P is parallel to the xy-plane. ✗ since $\langle 1, e, 0\rangle \cdot \langle 0,0,1\rangle = 0 \Rightarrow P \perp xy$-plane

(B) The line $x = 1$ is the intersection of P and the $\underbrace{xz\text{-plane}}_{y=0}$. ✓ if $y = 0$ then $x = 1$.

(C) The vector $\langle 1, e, 0\rangle$ is normal to P. ✓

(D) The point $(1, 0, e)$ is on the plane P. ✓

(E) P contains the line $x = 1 - e - et$, $y = t + 1$, $z = e$. ✓ $\rightsquigarrow$ verify that these parametric eqns satisfy the eqn of the tangent plane.

$\underbrace{1-e-et+e(t+1)}_{=0} \stackrel{?}{=} 0.$

First let's calculate the eqn of the tangent plane:

Let $F(x,y,z) = xz - \ln(y+e^z)$

$F_x(x,y,z) = z \Rightarrow F_x(1,0,-1) = -1.$

$F_y(x,y,z) = \frac{-1}{y+e^z} \Rightarrow F_y(1,0,-1) = -e.$

$F_z(x,y,z) = x - \frac{1}{y+e^z} e^z \Rightarrow F_z(1,0,-1) = 0.$

Then the eqn of the tangent plane is $x + ey = 0$.

10. The linearization $L(x, y, z)$ of $f(x, y, z) = xz^2 - yz + \cos(xy)$ at $(1, \pi, -1)$ is

$\Rightarrow$ (A) $x + y - (\pi + 2)z - \pi - 3$

(B) $x - y + 2z + \pi$

(C) $(1 - \pi)x - (1 + \pi)y + 2z + \pi + 3$

(D) $x + y - 2z + \pi + 3$

(E) $(1 - \pi)x - (1 + \pi)y - (\pi + 2)z$

$f(1,\pi,-1) = 1 + \pi - 1 = \pi$

$f_x(x,y,z) = z^2 - y\sin(xy) \Rightarrow f_x(1,\pi,-1) = 1.$

$f_y(x,y,z) = -z - x\sin(xy) \Rightarrow f_y(1,\pi,-1) = 1$

$f_z(x,y,z) = 2xz - y \Rightarrow f_z(1,\pi,-1) = -2-\pi.$

$L(x,y,z) = \pi + (x-1) + (y-\pi) - (\pi+2)(z+1)$

$= x + y - (\pi+2)z - \pi - 3$

Correct Choices for Multiple Choice Questions

Questions	MASTER	CODE 001	CODE 002	CODE 003	CODE 004
1	A	C	E	D	B
2	A	B	D	A	B
3	A	C	E	A	D
4	A	B	B	E	A
5	A	E	C	B	C
6	A	E	B	C	E
7	A	A	A	B	D
8	A	A	C	D	E
9	A	D	A	E	C
10	A	B	C	E	A