Q1) The matrix $A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ has only one eigen value $\lambda = 4$. Find the general solution of the system $X'(t) = AX(t)$.

Solution:

$A - 4I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Since A has only one eigenvalue (and A is a 3×3 matrix), $(A - 4I)^3$ must equal 0. Also

$$(A - 4I)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The fundamental solutions are

$$X_1(t) = e^{tA}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, X_2(t) = e^{tA}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, X_3(t) = e^{tA}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Using the formula – when A has only only one eigenvalue λ and $(A - \lambda I)^3 = 0$- that

$$e^{tA}v = e^{t\lambda}[v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2 v]$$

we get

$$X_1(t) = e^{4t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] = e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$X_2(t) = e^{4t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]$$

$$= e^{4t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= e^{4t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

$$X_3(t) = e^{4t}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$= e^{4t}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{t^2}{2}\ \begin{pmatrix}1\\0\\0\end{pmatrix}\right]$$

$$= e^{4t}\begin{pmatrix}t^2/2\\t\\1\end{pmatrix}$$

The general solution is $X(t) = c_1\, e^{4t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\, e^{4t}\begin{pmatrix}t\\1\\0\end{pmatrix} + c_3\, e^{4t}\begin{pmatrix}t^2/2\\t\\1\end{pmatrix}$

Note: If you follow the method in the book, the answer you will get is that the fundamental solutions are $X_1(t), X_1(t) + X_2(t), X_1(t) + X_2(t) + X_3(t)$.

Q2) The matrix $A = \begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}$ has complex eigenvalues $1 + 2i, 1 - 2i$. Find the general solution of the system $X^{'}(t) = AX(t)$.

Solution:

We can work with only one eigenvalue, say $1 + 2i$. The eigenvectors for this eigenvalue are given by

$$\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = (1 + 2i)\begin{pmatrix}x\\y\end{pmatrix}$$

So they are given by

$$x + 2y = (1 + 2i)x$$

$$-2x + y = (1 + 2i)y$$

Therefore the eigenvectors are given by $2y = 2ix$ and $-2x = 2iy$, that is $y = ix, -x = iy$.

This is really one equation $y = ix$. Thus $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\ix\end{pmatrix} = x\begin{pmatrix}1\\i\end{pmatrix}$

So a fundamental eigenvector is $\begin{pmatrix}1\\i\end{pmatrix}$. This gives a complex solution

$$X(t) = e^{tA}\begin{pmatrix}1\\i\end{pmatrix} = e^{t(1+2i)}\begin{pmatrix}1\\i\end{pmatrix}$$

We need to compute the real and imaginary parts of $X(t)$ to get two real fundamental solutions.

Now $e^{t(1+2i)} = e^{t+t2i} = e^t \ e^{2ti}$

Therefore $X(t) = \begin{pmatrix} e^t \ e^{2ti} 1 \\ e^t \ e^{2ti} i \end{pmatrix} = \begin{pmatrix} e^t \ (cos2t + isin2t \\ e^t \ (cos2t + isin2t)i \end{pmatrix} = \begin{pmatrix} e^t \ (cos2t + i \ sin2t \\ e^t \ (-sin2t + i \ cos2t) \end{pmatrix}$

The real part is $\begin{pmatrix} e^t \ cos2t \\ -e^t \ sin2t \end{pmatrix}$ and the imaginary part is $\begin{pmatrix} e^t \ sin2t \\ -e^t \ cos2t \end{pmatrix}$

Therefore the general solution is $c_1 \begin{pmatrix} e^t \ cos2t \\ -e^t \ sin2t \end{pmatrix} + c_2 \begin{pmatrix} e^t \ sin2t \\ -e^t \ cos2t \end{pmatrix}$

Note:

The components of the above vector give the general solution of the system of coupled differential equations

$$x'(t) = x(t) + 2y(t)$$

$$y'(t) = -2x(t) + y(t)$$

Q3) Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Solution:

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -1-\lambda & 0 & 1+\lambda \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$ (by adding -1R_3 to R_1)

$= (-1-\lambda) \begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} + (1+\lambda) \begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} = (-1-\lambda)[\lambda(\lambda-3)-4] + (1+\lambda)[4+4\lambda]$

$$= (-1-\lambda)[\lambda^2 - 3\lambda - 4] + (1+\lambda)(4+4\lambda)$$

$$= (-1-\lambda)(\lambda^2 - 3\lambda - 4 - 4 - 4\lambda)$$

$$= (-1-\lambda)(\lambda^2 - 7\lambda - 8)$$

$$= (-1-\lambda)(\lambda+1)(\lambda-8)$$

So eigen values are $-1, -1, 8$.

Note: This can, of course be done without any clever ideas. You will get a cubic equation. Its roots can be found by finding one rational root and by division- as in the problems in Chapter 4.