Exercise 1 (15 points 5-5-5): Let $R$ and $S$ be subrings of a ring $T$.
(1) Prove that $R \cap S$ is a subring of $T$.
(2) Under which condition $R \cup S$ is a subring of $T$?
(3) Find an example of a ring $T$ with two subrings $R$ and $S$ such that $R \cup S$ is not a subring of $T$. 
Exercise 2 (15 points 5-5-5): Let $R$ and $T$ be commutative rings, $f : R \rightarrow T$ be a ring homomorphism and $I$ an ideal of $R$.

(1) Prove that if $f$ is onto then $f(I)$ is an ideal of $T$.

(2) Prove that if $J$ is an ideal of $T$, then $E = \{ x \in R | f(x) \in J \}$ is an ideal of $R$.

(3) Find an example of commutative rings $R$ and $T$, $f : R \rightarrow T$ a ring homomorphism and $I$ an ideal of $R$ such that $f(I)$ is not an ideal of $T$. (Hint you may use $\mathbb{Z}, \mathbb{R}$ and any ideal of $\mathbb{Z}$).
**Exercise 3** (10 points 5-5): Let $R$ be a commutative ring with unity.

(1) Prove that if $R$ is of characteristic zero, then $\mathbb{Z}$ is isomorphic to a subring of $R$.

(2) Prove that if $R$ is of prime characteristic $p$, then $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to a subring of $R$. 
Exercise 4 (10 points 5-5): Let $R$ be a commutative ring and $M$ and $N$ two distinct maximal ideals of $R$.

(1) Prove that $M + N = R$, and $M \cap N = MN$.

(2) Give an example of a commutative ring with two maximal ideals $M$ and $N$ such that $M + N = R$. 
Exercise 5 (15 points, 5-5-5):
(1) Find all ring automorphisms of $\mathbb{Z}$.
(2) Find all ring automorphisms of $\mathbb{Q}$.
(3) Is there any ring isomorphism from $\mathbb{Z}$ to $\mathbb{Q}$, justify?
Exercise 6 (15 points, 5-5-5):
(1) Let \( m \) and \( n \) be two different positive integers. Prove that there is no isomorphism between \( m\mathbb{Z} \) and \( n\mathbb{Z} \).
(2) Prove that the quotient field \( K \) of an integral domain \( R \) is the intersection of all fields containing \( R \).
(3) What is the field of fractions of the ring \( R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \).