Exercise 1 (5 points)

By evaluating $\oint_C e^z \, dz$ around the unit circle $|z| = 1$, show that

$$\int_0^{2\pi} e^{\cos \theta} \cos(\theta + \sin \theta) \, d\theta = \int_0^{2\pi} e^{\cos \theta} \sin(\theta + \sin \theta) \, d\theta = 0.$$
Exercise 2 (10 points)

Let \( f(z) = \frac{z^2 + 2z - 5}{(z^2 + 4)(z^2 + 2z + 2)} \).

(a) If \( C_R \) is the circle \(|z| = R\), show that \( \lim_{R \to +\infty} \oint_{C_R} f(z) \, dz = 0 \).

(b) Use the result (a) to deduce that if \( C \) is the circle \(|z - 2| = 5\), then \( \oint_C f(z) \, dz = 0 \).

(c) Compute \( \oint_C f(z) \), where \( C \) is the circle \(|z + 1| = 2\) traversed once the positive sens.
Exercise 3 (7 points)
Let \( C_r \) be the circle \(|z| = r\) traversed once the positive sens. Find for \( r > 0 \) and \( r \neq 1 \), the integral

\[
\oint_{C_r} \frac{z}{(1-z)^2} \, dz.
\]
Exercise 4 (6 points) Show that

\[ \int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2} (1 - i), \]

where the integrand denotes the principal branch

\[ z^i = \exp(i \log z) \quad (|z| > 0, -\pi < \text{Arg} z < \pi) \]

of \( z^i \) and where the path of integration is any contour from \( z = -1 \) to \( z = 1 \) that, except for its end points, lies above the real axis.
Exercise 5 (10 points)

(a) Let \( f \) be an entire and suppose that \( \Re f(z) \leq M \) for all \( z \). Prove that \( f \) must be a constant function. [Hint: Consider the function \( e^f \).]

(b) Suppose that \( f \) is entire and that \( |f(z)| \leq |z|^3 \) for all sufficiently large values of \( |z| \). Prove that \( f \) must be a polynomial of degree at most 3.
Exercise 6 (5 points)
Let $f_n$ be a sequence of functions analytic in a simply connected domain $D$ and converging uniformly to $f$ in $D$. Prove that $f$ is analytic in $D$. 
Exercise 7 (7 points)

Find the Laurent series for the function $\frac{z^2}{(z - 1)(z + 2)}$ in each of the following domains

(a) $|z| < 1$  
(b) $1 < |z| < 2$  
(c) $|z| > 2$.  
