King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH430 - Introduction to Complex Variables
Exam II – Term 132 (2013–2014)

Exercise 1 (5 points)

By evaluating $\oint_C e^z\,dz$ around the unit circle $|z| = 1$, show that

$$\int_0^{2\pi} e^{\cos\theta} \cos(\theta + \sin\theta)\,d\theta = \int_0^{2\pi} e^{\cos\theta} \sin(\theta + \sin\theta)\,d\theta = 0.$$

Exercise 2 (10 points)

Let $f(z) = \dfrac{z^2 + 2z - 5}{(z^2 + 4)(z^2 + 2z + 2)}$.

(a) If $\mathcal{C}_R$ is the circle $|z| = R$, show that $\lim_{R \to +\infty} \oint_{\mathcal{C}_R} f(z)\, dz = 0$.

(b) Use the result (a) to deduce that if $\mathcal{C}$ is the circle $|z - 2| = 5$, then $\oint_{\mathcal{C}} f(z)\, dz = 0$.

(c) Compute $\oint_{\mathcal{C}} f(z)$, where $\mathcal{C}$ is the circle $|z + 1| = 2$ traversed once the positive sens.

Exercise 3 (7 points)
Let $\mathcal{C}_r$ be the circle $|z| = r$ traversed once the positive sens. Find for $r > 0$ and $r \neq 1$, the integral

$$\oint_{\mathcal{C}_r} \frac{\overline{z}}{(1-z)^2}\, dz.$$

Exercise 4 (6 points) Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2}(1 - i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis.

Exercise 5 (10 points)

(a) Let f be an entire and suppose that $\Re f(z) \leq M$ for all z. Prove that f must be a constant function. [HINT: Consider the function e^f.]

(b) Suppose that f is entire and that $|f(z)| \leq |z|^3$ for all sufficiently large values of $|z|$. Prove that f must be a polynomial of degree at most 3.

Exercise 6 (5 points)

Let f_n be a sequence of functions analytic in a simply connected domain D and converging uniformly to f in D. Prove that f is analytic in D.

Exercise 7 (7 points)

Find the Laurent series for the function $\frac{z^2}{(z-1)(z+2)}$ in each of the following domains

$(a)\ |z|<1$ $\qquad (b)\ 1<|z|<2$ $\qquad (c)\ |z|>2.$