Name: ___________________________ ID#: __________________
Instructor: ______________________ Sec #: ______ Serial #: ______

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

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Q:1 (16 points) Solve the heat equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \]

subject to the following initial and nonhomogeneous boundary conditions

\[ u(x,0) = 4 \text{ for } 0 < x < \pi \text{ and } u(0,t) = 0, \quad u(\pi,t) = 4 \text{ for } t > 0. \]
Q:2 (16 points) Use Laplace transformation method to solve the wave equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = te^{-x}, \quad 0 < x < \infty, \quad t > 0,
\]

with the initial conditions \(u(x, 0) = 0, \ u_t(x, 0) = x,\ \text{for} \ 0 < x < \infty\)

and the boundary conditions \(u(0, t) = 1 - e^{-t}, \ \lim_{x \to \infty} |u(x, t)| \sim x^n,\ \text{for some finite} \ n, \ t > 0.\)
Q:3 (14 points) Find steady-state temperature in a semi infinite plate by solving
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0
\]
subject to the following boundary conditions \( u(0, y) = 0, \ u(\pi, y) = 0 \) for \( y > 0 \)
and \( u(x, 0) = x, \ 0 < x < \pi \). Also solution is bounded at \( y \to \infty \).
Q:4 (20 points) Find the steady-state temperature in a hemisphere of radius 2 by solving
\[
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \quad 0 < r < 2, \quad 0 < \theta < \frac{\pi}{2}
\]
when the base of the hemisphere is insulated \([u_{\theta}(r, \frac{\pi}{2}) = 0]\)

and \(u(2, \theta) = \sin(\theta) , \quad 0 < \theta < \frac{\pi}{2}\). Find first three nonzero terms of the series solution.

(Hint: \(P_n'(0) = 0\) only for even values of \(n\))
Q:5 (20 points) Find the displacement \( u(x, t) \) in a circular plate of radius 2 by solving
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 2, \quad t > 0
\]
with initial conditions \( u(r, 0) = r^2, \) \( u_t(r, 0) = 0, \) \( 0 < r < 2, \)
and the boundary condition \( u(2, t) = 0, \) \( t > 0. \) Solution is bounded at \( r = 0. \)
Q:6 (14 points) Solve the nonhomogeneous linear system using variation of parameters

\[
X' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} e^t \cos t \\ e^t \sin t \end{bmatrix}, \text{ with } X(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]