King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Final Exam for Math 572, Semester 132

Name:
ID:

Problem 1 Consider the following two dimensional mixed problem:

$$\begin{cases} -\mathrm{div}(a\nabla u) + c\,u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \partial_n\, u = g & \text{on } \Gamma_2, \end{cases} \tag{1}$$

where the functions a, c and f are sufficiently regular with $a(x) \geq a_0 > 0$ and $c(x) \geq 0$ for $x \in \Omega$.

a) Define the weak formulation of the mixed problem (1) on a suitable Sobolev space $\mathbf{H}$ (You need to define $\mathbf{H}$).

b) Show that the mixed problem (1) has a unique weak solution $u \in \mathbf{H}$.

d) Define the continuous piecewise bilinear finite element solution u_h of (1) over a uniform mesh consists of squares with side length equal to h.

e) Prove the existence and uniqueness of u_h.

f) Show that $\|u - u_h\|_{H^1} \leq C\,h\|u\|_{H^2}$.

Problem 2 Consider the following problem:

$$\begin{cases} u_t - u_{xx} = u\sin(u) & \text{on } \ \Omega \times (0,T] \\ u(0,t) = u(1,t) = 0 & \text{for } \ t \in (0,T) \\ u(x,0) = v(x) & \text{for } \ x \in \Omega \end{cases} \tag{2}$$

where $\Omega = (0,1)$ and $u = u(x,t)$. Assume that $u \in \mathcal{C}^1([0,T]; H^2(\Omega))$.

a) Show that $\|u(t)\| \le C\|v\|$ for any $t \in (0,T]$.

b) Discretize problem (2) in time by using (first order) backward Euler scheme over a uniform mesh consists of N subintervals and each is of length k.

c) Say that $U^n \approx u(t_n)$ is the backward Euler solution, for $n = 1, \cdots, N$, and assume that the time-step size k is sufficiently small. Prove the following stability property:

$$\|U^n\| \le C\|v\| \quad \text{for} \ \ n = 1, \cdots, N.$$

d) Show that $\|U^n - u(t_n)\| \le C(u)\, k$ for $n = 1, 2, \cdots, N$.

Problem 3 Consider the following first order model:

$$u_t = a\,u_x \quad \text{in } \mathbb{R}\times\mathbb{R}_+ \quad \text{with } u(x,0) = v(x) \quad \text{for } x\in\mathbb{R}, \tag{3}$$

where a is a positive constant and v is a smooth function.

Define the grid points:

$$x_m = mh \ \text{ for } \ m\in\mathbb{Z} \quad \text{and} \quad t_n = nk \ \text{ for } \ n\in\mathbb{N},$$

where h and k are the mesh step sizes in space and time, respectively.
Introduce the grid functions

$$U_m^n \approx u(x_m,t_n) \ \text{ and } \ U_m^0 = v_m = v(x_m).$$

where U_m^n is defined through the finite difference (FD) scheme:

$$\frac{U_m^{n+1}-U_m^n}{k} = a\frac{U_{m+1}^n - U_m^n}{h} \quad \text{for } m\in\mathbb{Z} \text{ and } n\in\mathbb{N}$$
$$U_m^0 = v_m \quad \text{for } m\in\mathbb{Z}$$

a) Set $\lambda = \frac{k}{h}$. Show that

$$U_m^n = a\lambda U_{m+1}^n + (1-a\lambda)U_m^n$$

Assume that $a\lambda \le 1$. Then, for $n = 0,1,2,3,\cdots$, prove that

b)

$$\max_{m\in\mathbb{Z}} |U_m^n| \le C\|v\|_{\mathcal{C}}$$

c)

$$\max_{m\in\mathbb{Z}} |U_m^n - u(x_m,t_n)| \le C t_n h\,\|v\|_{\mathcal{C}^2}$$