MATH 590.2 (Term 132)
Final Exam

May 22, 2014

Problem 1. Let Ω be an open bounded set of $\mathbb{R}^d$, $d = 1, 2$, or 3, with smooth boundary $\partial\Omega$. Consider the initial and boundary value problem (this is the viscous Cahn-Hilliard equation)

$$\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1}$$

$$u|_{t=0} = u_0, \qquad \text{in } \Omega, \tag{2}$$

$$\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, \qquad \text{on } \partial\Omega \times \mathbb{R}_+, \tag{3}$$

where n is the unit outward normal on $\partial\Omega$, $u(x,t)$, $x \in \Omega$, $t \in \mathbb{R}_+$, is the unknown scalar function, $f \in \mathcal{C}^1(\mathbb{R})$ is a function satisfying the conditions:

$$|f'(s)| \leq c_1(|s|^2 + 1), \ \ \forall s \in \mathbb{R}, \tag{4}$$

$$sf(s) \geq F(s) - c_2, \ \ \forall s \in \mathbb{R}, \tag{5}$$

$$f'(s) \geq -c_3, \ \ \forall s \in \mathbb{R}, \tag{6}$$

$$\text{for every } \epsilon > 0, \text{ there exists } \mathrm{C}_\epsilon > 0 \text{ such that } |f(s)| \leq \epsilon F(s) - C_\epsilon, \ \ \forall s \in \mathbb{R}, \tag{7}$$

$$F(s) \geq -c_4, \ \ \forall s \in \mathbb{R}, \tag{8}$$

where $F(s) = \int_0^s f(\tau)d\tau$.

We introduce the space

$$V = \{u \in H^2(\Omega), \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}.$$

1.) Implement Galerkin approximation method to prove that, if $u_0 \in H^1(\Omega)$, then problem (1)-(3) possesses a unique solution u which satisfies

$$u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap L^2(0,T;V), \quad u_t \in L^2(0,T;L^2(\Omega)), \quad \forall T > 0.$$

2.) Prove that, if $u_0 \in V$, then

$$u \in L^\infty(0,T;V), \quad u_t \in L^2(0,T;H^1(\Omega)), \quad \forall T > 0.$$

Solution: If X is a Sobolev-type space, then we set

$$\dot{X} = \{q \in X, \, m(q) = 0\},$$

where

$$m(q) = \frac{1}{L}\int_\Omega q(x)\,dx.$$

Moreover, we set

$$\bar{q} = q - m(q).$$

Let us define the linear unbounded operator operator

$$-\Delta : V \to \dot{H}$$

which is self-adjoint and nonnegative. If $-\Delta$ is restricted to $\dot{V}$, then it turns to be positive with compact inverse $(-\Delta)^{-1}$. There is a complete orthonormal family w_j on $L^2(\Omega)$ made of eigenvectors of $-\Delta$ such that

$$-\Delta w_j = \lambda_j w_j, \quad w_j \in V,$$

(in fact, $w_j \in W$, since $\partial\Omega$ is sufficiently regular) and

$$0 = \lambda_1 < \lambda_2 < ... < \lambda_m < ..$$

For every $r > 0$, we endow $(D(-\Delta)^{r/2})'$ with the norm $\|q\|_{-r} = (\|(-\Delta)^{-r/2}\bar{q}\|^2 + |m(q)|^2)^{1/2}$. We also note that $\|q\|_r = (\|(-\Delta)^{r/2}q\|^2 + |m(q)|^2)^{1/2}$ is a norm on $D((-\Delta)^{r/2})$ which is equivalent to the usual $H^r(\Omega)-$norm

1.)a.) Weak formulation: For any given $T > 0$, find $u : [0,T] \to V$ such that

$$u(0) = u_0,$$

and, for almost every $t \in [0,T]$,

$$\frac{d}{dt}[(u,q) + (\nabla u, \nabla q)] + (\Delta u, \Delta q) - (f'(u)\nabla u, \nabla q) = 0, \ \ \forall q \in V. \tag{9}$$

b.) Let take $q = 1$ (which belongs to V) in (9). We deduce that

$$\frac{d}{dt}\int_\Omega u(x,t)dx = 0. \tag{10}$$

We integrate (10) between 0 and t, and we find

$$\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx, \quad \forall t \geq 0. \tag{11}$$

c.) We set $E_m = \text{span}\{w_1, ..., w_m\}$ and P_m is the orthogonal projection on E_m. We consider the approximate problem:
For each m, find an approximate solution u_m of the form $u_m(t) = \sum_{j=1}^m c_j(t)w_j$ satisfying

$$\frac{d}{dt}[(u_m, w_j) + (\nabla u_m, \nabla w_j)] + (\Delta u_m, \Delta w_j) + (f'(u_m)\nabla u_m, \nabla w_j) = 0, \quad j = 1, ..., m, \tag{12}$$

$$u_m(0) = P_m u_0, \tag{13}$$

d.) Problem (12)-(13) yields

$$(1 + \lambda_j)c_j' + c_j\lambda_j^2 + (f'(u_m)\nabla u_m, \nabla w_j) = 0, \quad j = 1, ..., m. \tag{14}$$

Now, setting $Y = (c_1, \ldots, c_m)$ we infer from (14) a system of the form:

$$M_1Y' + M_2Y + F(Y) = 0, \tag{15}$$

where

$$M_1 = ((1+\lambda_i)\delta_{ij})_{i,j=1,\dots,m}\,, \quad M_2 = \left(\lambda_i^2\delta_{ij}\right)_{i,j=1,\dots,m}$$

and

$$F(Y) = ((f'(\sum_{j=1}^{m} c_j w_j)\sum_{j=1}^{m} c_j\nabla w_j, \nabla w_1), \dots, (f'(\sum_{j=1}^{m} c_j w_j)\sum_{j=1}^{m} c_j\nabla w_j, \nabla w_m)).$$

The matrix M_1 is invertible and $F(Y)$ depends continuously on Y. Hence, (12)-(13) has a unique solution on some finite time interval $[0, T_m[$.
e.) Now, we have to show that the solutions are bounded in time and uniformly bounded in m. First we note that (12) is also equivalent to

$$\frac{du_m}{dt} - \Delta\frac{du_m}{dt} + \Delta^2 u_m - P_m\Delta f(u_m) = 0, \tag{16}$$

We multiply (16) by u_m, integrate over Ω and we find

$$\frac{1}{2}\frac{d}{dt}[\|u_m\|^2 + \|\nabla u_m\|^2] + \|\Delta u_m\|^2 + \int_\Omega f'(u_m)|\nabla u_m|^2 dx = 0.$$

Using (6), we deduce

$$\frac{1}{2}\frac{d}{dt}[\|u_m\|^2 + \|\nabla u_m\|^2] + \|\Delta u_m\|^2 \le c_{10}\|\nabla u_m\|^2, \tag{17}$$

Neglecting the two terms in the middle, we find

$$\frac{d}{dt}[\|u_m\|^2 + \|\nabla u_m\|^2] \le c(\|u_m\|^2 + \|\nabla u_m\|^2], \tag{18}$$

We apply the Gronwall lemma to (18), and noting that $\|u_m(0)\|^2 + \|\nabla u_m(0)(t)\|^2 \le \|u_0\|^2 + \|\nabla u_0\|^2$, we get

$$\|u_m(t)\|^2 + \|\nabla u_m\|^2 \le c[\|u_0\|^2 + \|\nabla u_0\|^2]e^{ct}, \quad \forall t \ge 0,$$

hence

$$\sup_{t\in[0,T]} \|u_m(t)\|^2_{H^1} \le C_1. \tag{19}$$

Now, we add $m(u_m)^2$ in both sides of (17), and we integrate the resulting equation between 0 and T, and we find

$$\begin{aligned}\|u_m(T)\|^2 + \|\nabla u_m(T)\|^2 + \int_0^T (|m(u_m)|^2 + \|\Delta u_m\|^2)dt \le & c\int_0^T (|m(u_0)|^2 + \|\nabla u_m\|^2)dt \\ \le & C(T),\end{aligned}$$

It follows that

$$\int_0^T \|u_m\|^2_{H^2} dt \le C(T). \tag{20}$$

Lastly, we deduce from (16) that

$$\frac{du_m}{dt} = -(I-\Delta)^{-1}\Delta[\Delta u_m - f(u_m)]. \tag{21}$$

We have that

$$\Delta u_m \in L^2(0,T;L^2(\Omega)).$$

We have

$$\begin{aligned}\int_0^T \|f(u_m)\|^2 dt &\leq \int_0^T \int_\Omega |f(u_m)|^2 dx dt \\ &\leq c_3 \int_0^T \int_\Omega u_m^6 dx dx + c_4 T \\ &\leq c_3 \int_0^T \|u_m\|^6_{L^6(\Omega)} dt + c_4 T \\ &\leq c_3 \int_0^T \|u_m\|^6_{H^1(\Omega)} dt + c_4 T \\ &\leq C(T),\end{aligned} \tag{22}$$

on account of (19). From the regularity of elliptic problems, we know that

$$\|(I-\Delta)^{-1}u\|_{H^2} \leq c\|u\|.$$

Therefore

$$\|(I-\Delta)^{-1}\Delta u\| \leq \|(I-\Delta)^{-1}u\|_{H^2} \leq c\|u\|$$

We can deduce from (21) that

$$\|\frac{du_m}{dt}\| \leq c\|\Delta u_m - f(u_m)\|, \tag{23}$$

and therefore

$$\begin{aligned}\int_0^T \|\frac{du_m}{dt}\|^2 dt \leq & c\int_0^T \|\Delta u_m\|^2 dt + \int_0^T \|f(u_m)\|^2 dt \\ & C(T).\end{aligned} \tag{24}$$

We could also have proceeded as follows. Since $u_m \in L^2(0,T;V)$, we have that $\Delta^2 u_m \in L^2(0,T;V')$. On the other hand, since $f(u_m) \in L^2(0,T;L^2(\Omega)$, it follows that $-\Delta f(u_m) \in L^2(0,T;V')$. We deduce from (16) that $(I-\Delta)\frac{du_m}{dt} = -\Delta^2 u_m + \Delta f(u)$, and consequently that $(I-\Delta)\frac{du_m}{dt} \in L^2(0,T;V')$, hence $\frac{du_m}{dt} \in L^2(0,T;L^2(\Omega))$.

The constants from C_1 through C_4 are all bounded for bounded sets of initial conditions and bounded time intervals. These estimates are also bounded uniformly in m. This yields that the solution is global (i.e. $T_m = T$). We have the existence of a convergent subsequence in the respective Sobolev spaces and we can pass to the limit. The fact that $u \in L^2(0,T,V)$ and $\frac{du}{dt} \in L^2(0,T;L^2\Omega))$ implies that $u \in \mathcal{C}([0,T];L^2(\Omega))$. The fact that $u \in L^\infty(0,T,H^1(\Omega))$ implies that $u \in \mathcal{C}_w([0,T];H^1(\Omega))$. The fact that the map $[0,T] \to H^1(\Omega)$, $t \mapsto \|\nabla u(t)\|$ is continuous implies that $u \in \mathcal{C}([0,T]; H^1(\Omega))$.

Indeed, we observe that, for $u \in V$ and $g \in L^2(\Omega)$, we have

$$-\Delta u = \bar{g} \quad \Longleftrightarrow \quad u = (-\Delta)^{-1}\bar{g}.$$

Thus, the equation (16) can be rewritten as

$$-\Delta \left[\frac{d}{dt}(-\Delta)^{-1}\bar{u}_m + \frac{du_m}{dt} - \Delta u_m + P_m f(u_m) \right] = 0,$$

which is equivalent to

$$\frac{d}{dt}[(-\Delta)^{-1}\bar{u}_m + u_m] - \Delta u_m + P_m f(u_m) = 0. \tag{25}$$

(since (-Δ)$^{-1}0 = 0$).
We multiply (25) by $\frac{du_m}{dt}$, integrate over Ω and we find

$$\frac{1}{2}\frac{d}{dt}[\|\nabla u_m\|^2] + \|\frac{du_m}{dt}\|^2 + \|\frac{du_m}{dt}\|^2_{-1} + \int_\Omega f(u_m)\frac{du_m}{dt}dx = 0.$$

We integrate between t_0 and t, and we find

$$\|\nabla u_m(t)\|^2 + 2\int_{t_0}^t \|\frac{du_m}{dt}\|^2 ds + 2\int_{t_0}^t \|\frac{du_m}{dt}\|^2_{-1} ds + 2\int_{t_0}^t (f(u_m), \frac{du_m}{dt})ds = \|\nabla u_m(t_0)\|^2.$$

All the integral terms tend to zero when t tends to t_0, hence

$$\lim_{t\to t_0} \|\nabla u_m(t)\|^2 = \|\nabla u_m(t_0)\|^2.$$

Uniqueness: Let $w = u_1 - u_2$, where u_1 and u_2 are two solutions of (1)-(3). There holds $w(0) = 0$ and $m(w) = 0$. We multiply (25) by w, and we integrate over Ω, and we find

$$\frac{1}{2}\frac{d}{dt}[\|w\|^2_{-1} + \|w\|^2] + \|\nabla w\|^2 + \int_\Omega (f(u_1) - f(u_2))w dx = 0. \tag{26}$$

We have $f(u_1) - f(u_2) = f'(c)w$, with $c = \theta u_1 + (1-\theta)u_2$, $\theta \in (0,1)$; hence

$$\begin{aligned}\int_\Omega (f(u_1) - f(u_2))w dx &= \int_\Omega f'(c)|w|^2 dx \\ &\geq -C\|w\|^2.\end{aligned}$$

We deduce from (26) that

$$\frac{1}{2}\frac{d}{dt}[\|w\|^2_{-1} + \|w\|^2] + \|\nabla w\|^2 \leq c\|w\|^2,$$

and then

$$\frac{d}{dt}[\|w\|^2_{-1} + \|w\|^2] \leq c(\|w\|^2_{-1} + +\|w\|^2). \tag{27}$$

We multiply by e^{-ct} and integrate between 0 and t, and we find

$$\|w(t)\|_{-1}^2 + \|w(t)\|^2 \le e^{ct}(\|w(0)\|_{-1}^2 + \|w(0)\|^2), \quad \forall t \ge 0;$$

hence $w(t) = 0$, and $u_1(t) = u_2(t)$.

2.) We multiply (16) by $\frac{du_m}{dt}$, integrate over Ω and we find

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_m\|^2 + \|\frac{du_m}{dt}\|^2 + \|\nabla \frac{du_m}{dt}\|^2 - \int_\Omega \nabla f(u_m)\nabla \frac{du_m}{dt}dx = 0.$$

We have

$$\begin{aligned} |\int_\Omega \nabla f(u_m)\nabla \frac{du_m}{dt}dx| &\le \int_\Omega |f'(u_m)||\nabla u_m||\nabla \frac{du_m}{dt}|dx \\ &\le \|f'(u_m)\|_{L^3}\|\nabla u_m\|_{L^6}\|\nabla \frac{du_m}{dt}\| \\ &\le c(\|u_m\|_{L^6}^2 + 1)\|\nabla u_m\|_{L^6}\|\nabla \frac{du_m}{dt}\| \\ &\le c(\|u_m\|_{H^1}^2 + 1)\|u_m\|_{H^2}\|\nabla \frac{du_m}{dt}\|. \end{aligned}$$

We then get and then

$$\frac{d}{dt}\|\Delta u_m\|^2 + \|\frac{du_m}{dt}\|^2 + \frac{1}{2}\|\nabla \frac{du_m}{dt}\|^2 \le c(\|u_m\|_{H^1}^2 + 1)^2\|u_m\|_{H^2}^2. \tag{28}$$

Noting that

$$\|u_m\|_{H^2}^2 \le c(|m(u_m)|^2 + \|\Delta u_m\|^2),$$

we can deduce from (28) that

$$\sup_{t\in[0,T]} \|u\|_{H^2}^2 \le C(T), \tag{29}$$

and

$$\int_0^T \|\frac{du_m}{dt}\|_{H^1}^2 dt \le C(T). \tag{30}$$

The constants from C_5 through C_7 are all bounded for bounded sets of initial conditions and bounded time intervals. These estimates are also bounded uniformly in m. We have the existence of a convergent subsequence in the respective Sobolev spaces and we can pass to the limit.

Problem 2. We introduce the spaces

$$H_\beta = \left\{ q \in H^1(\Omega),\ \frac{1}{|\Omega|}\int_\Omega q(x)dx = \beta \right\}, \quad \beta \in \mathbb{R},$$
$$\mathscr{H}_\alpha = \bigcup_{|\beta|\leq\alpha} H_\beta.$$

We define the semigroup

$$S(t):\ \ H^1(\Omega) \to H^1(\Omega),$$
$$u_0 \mapsto u(t),$$

where $u(t)$ is the solution of (1)-(3) (see Problem 1) at time t.

1.) Show that $S(t)$ has an absorbing set B_1 in $\mathscr{H}_\alpha$.
3.) Show that $S(t)$ has an absorbing set B_2 in $\mathscr{H}_\alpha \cap H^2(\Omega)$.
4.) Deduce that, for every $\alpha \geq 0$, $S(t)$ has a global attractor $\mathcal{A}_\alpha$ in $\mathscr{H}_\alpha$.

Solution:
1.) let

$$u_0 \in B = \{u \in H^1(\Omega),\ \|u\|_{H^1} \leq R,\ |m(u)| \leq \alpha\}$$

for some $R > 0$. We saw that (1) can be reduced to

$$\frac{d}{dt}[(-\Delta)^{-1}\bar{u} + u] - \Delta u + f(u) = 0. \tag{31}$$

Now, we multiply (31) by $\bar{u}$ and integrate over Ω, and we find

$$\frac{1}{2}\frac{d}{dt}[\|\bar{u}\|_{-1}^2 + \|\bar{u}\|^2] + \|\nabla u\|^2 + \int_\Omega f(u)udx - m(u)\int_\Omega f(u)dx = 0.$$

Using Assumption (5), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|\bar{u}\|_{-1}^2 + \|\bar{u}\|^2] + \|\nabla u\|^2 + c\int_\Omega F(u)dx \leq & |m(u_0)|\int_\Omega |f(u)|dx + c_2|\Omega| \\ \leq & \alpha\int_\Omega |f(u)|dx + c_2|\Omega|.\end{aligned}$$

Using now assumption (7), we deduce that

$$\frac{1}{2}\frac{d}{dt}[\|\bar{u}_m\|_{-1}^2 + \|\bar{u}_m\|^2] + \|\nabla u_m\|^2 + c\int_\Omega F(u)dx \leq c_1(\alpha). \tag{32}$$

We multiply (1) by $\frac{du}{dt}$, integrate over Ω and we find

$$\frac{1}{2}\frac{d}{dt}[\|\nabla u\|^2 + 2\int_\Omega F(u)dx] + \|\frac{du}{dt}\|^2 + \|\frac{du}{dt}\|_{-1}^2 = 0. \tag{33}$$

Summing (32) and (33), we get

$$\frac{1}{2}\frac{d}{dt}E_1(t) + \|\frac{du}{dt}\|^2 + \|\frac{du}{dt}\|_{-1}^2 + \|\nabla u_m\|^2 + c\int_\Omega F(u)dx \leq c_1(\alpha), \tag{34}$$

where

$$E_1(t) = \|\bar{u}_m\|_{-1}^2 + \|\bar{u}_m\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)dx$$

We get from the poincaré inequality that

$$\|\bar{u}\| \leq c\|\nabla u\|, \quad \forall u \in H^1(\Omega).$$

Since $L^2(\Omega) \subset H^{-1}(\Omega)$, we also have

$$\|\bar{u}\|_{-1} \leq c\|\bar{u}\|, \quad \forall u \in L^2(\Omega).$$

There exists $c_0 > 0$ such that

$$\|\nabla u_m\|^2 + c\int_\Omega F(u)dx \geq c_0 E_1(t)$$

We then get from (34) that

$$\frac{d}{dt}E_1(t) + c_0 E_1(t) \leq c_3(\alpha). \tag{35}$$

Applying the Gronwall'lemma, we find

$$E_1(t) \leq E_1(0)e^{-c_0 t} + c_3(\alpha)(1 - e^{-c_0 t}), \ \ \forall t \geq 0.$$

WE now note that there exist $c_1, c_2 > 0$ such that

$$\|\nabla u\|^2 - c_1 \leq E_1(t) \leq c_2(\|u\|_{H^1}^4 + 1).$$

Thus,

$$B_1 = \{u \in H^1(\Omega), \ |m(u)|^2 + \|\nabla u\|^2 \leq 2c_3(\alpha) + c_1 + \alpha^2 = \varrho_1^2\}$$

is an absorbing set for $S(t)$.

2.) We recall from (28) Problem 1 that

$$\frac{d}{dt}\|\Delta u\|^2 \leq c(\|u\|_{H^1}^2 + 1)^2\|u\|_{H^2}^2 \tag{36}$$

There exists $t(R) > 0$ such that

$$|m(u(t))|^2 + \|\nabla u(t)\|^2 \leq \varrho_1^2, \ \ \forall t \geq t(R) = t_1.$$

We integrate (35) between t and $t+1$, for any $t \geq t_1$, we deduce

$$c_0\int_t^{t+1} E_1(s)ds \leq E_1(t) + c_3(\alpha), \ \ \forall t \geq t_1,$$

hence

$$\int_t^{t+1} \|u\|_{H^1}^2 ds \leq c_4(\varrho_1), \ \ \forall t \geq t_1.$$

Now, we integrate (17) between t_1 and t, we deduce

$$\int_t^{t+1} \|u\|_{H^2}^2 ds \leq c_5(\varrho_1), \ \ \forall t \geq t_1.$$

We deduce from (36) that

$$\frac{d}{dt}\|\Delta u\|^2 \leq c_6(\varrho_1)\|\Delta u\|^2 + c_7(\varrho_1), \ \ \forall t \geq t_1 \tag{37}$$

WE now apply the uniform Gronwall lemma to (37), and we find that

$$\|\Delta u(t)\|^2 \leq c_8(\varrho_1), \ \ \forall t \geq t_1 + 1$$

Thus,

$$B_2 = \{u \in H^2(\Omega), \ |m(u)|^2 + \|\Delta u\|^2 \leq c_8(\varrho_1) + \alpha^2 = \varrho_2^2\}$$

is an absorbing set for $S(t)$.

3.) The semigroup $S(t)$ is uniformly compact and has a global attractor $\mathcal{A}$ in $\mathscr{H}_\alpha$.

Problem 3. Let Ω be an open bounded set of $\mathbb{R}^d$, $d \leq 3$, with smooth boundary $\partial\Omega$. Consider the initial and boundary value problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) + p(u) = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{38}$$

$$u(x,0) = u_0, \quad \frac{\partial u}{\partial t}(x,0) = u_1, \quad \text{in } \Omega, \tag{39}$$

$$u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}_+, \tag{40}$$

where $u(x,t)$, $x \in \Omega$, $t \in \mathbb{R}_+$, is the unknown scalar function. We assume that $f, p \in \mathcal{C}^1(\mathbb{R})$ and

$$F(s) \geq -c_1, \ \forall s \in \mathbb{R}, \tag{41}$$

$$sf(s) - c_2 F(s) \geq -c_3, \ \forall s \in \mathbb{R}, \tag{42}$$

$$|f'(s)| \leq c_4(|s|^2 + 1), \quad \forall s \in \mathbb{R}, \tag{43}$$

$$|p(s)| \leq c_5 (F(s) + 1)^{\frac{1}{2} - \sigma_1}, \quad \sigma_1 > 0, \tag{44}$$

and

$$\begin{aligned} &\text{if } \|\nabla u\| \leq R, \text{ then there exists } C(R) > 0 \text{ such that} \\ &\|\nabla(f(u) + p(s))\| \leq C(R)(1 + \|\Delta u\|)^{1-\sigma_2}, \quad \sigma_2 > 0, \end{aligned} \tag{45}$$

where $F(s) = \int_0^s f(\varsigma) d\varsigma$.

Let introduce the spaces

$$E_1 = H_0^1(\Omega) \times L^2(\Omega), \quad E_2 = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega).$$

We can define the semigroup

$$\begin{aligned} S(t): \ E_j \to E_j, \ \ j = 1, 2, , \\ (u_0, u_1) \mapsto (u(t), u_t(t)), \end{aligned}$$

where $u(t)$ is the solution of (38)-(40) at time t.
1.) Show that $S(t)$ has an absorbing set B_1 in E_1.
2.) Show that $S(t)$ has an absorbing set B_2 in E_2.

Solution:
1.) Let

$$(u_0, u_1) \in B = \{(u, v) \in E_1, \ \|(u, v)\|_{E_1} \leq R\},$$

for some $R > 0$. We multiply (38) by u, and we integrate over Ω, and we obtain

$$\frac{d}{dt}(u_t, u) + \|\nabla u\|^2 + (u, u_t) + (f(u), u) + (p(u), u) = \|u_t\|^2. \tag{46}$$

Using assumption (42), we have

$$\int_\Omega f(u)udx \geq c_1 \int_\Omega F(u)dx - C.$$

We infer from (46) that

$$\begin{aligned}\frac{d}{dt}(u_t,u) + \|\nabla u\|^2 + (u,u_t) + c_1\int_\Omega F(u)dx \leq & \|u_t\|^2 + \|u\|\|p(u)\| \\ & \leq \|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + c\|p(u)\|^2,\end{aligned}$$

hence

$$\frac{d}{dt}(u_t,u) + \frac{1}{2}\|\nabla u\|^2 + (u,u_t) + c_1\int_\Omega F(u)dx \leq \|u_t\|^2 + c\|p(u)\|^2. \tag{47}$$

We multiply (38) by u_t, and we integrate over Ω, and we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)dx\right] + \|u_t\|^2 + (p(u),u_t) = 0. \tag{48}$$

We then deduce that

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)dx\right] + \|u_t\|^2 \leq & \|p(u)\|\|u_t\| \\ & \leq \frac{1}{2}\|u_t\|^2 + c\|p(u)\|^2,\end{aligned}$$

hence

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)dx\right] + \|u_t\|^2 \leq c\|p(u)\|^2, \tag{49}$$

Summing ϖ(47) and (49), for some small $\varpi \in (0,1)$, we deduce

$$\frac{d}{dt}E(t) + (1-\varpi)\|u_t\|^2 + \varpi(\|\nabla u\|^2 + (u,u_t) + \int_\Omega F(u)dx) \leq c\|p(u)\|^2, \tag{50}$$

where

$$E(t) = \varpi(u_t,u) + \left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)dx\right].$$

Finally, we have

$$\begin{aligned}\|p(u)\|^2 \leq & c_2\int_\Omega (F(s)+1)^{1-2\sigma_2}dx \\ & \leq \epsilon\int_\Omega F(s)dx + c_\epsilon, \ \forall \epsilon > 0.\end{aligned}$$

We deduce from (50) that

$$\frac{d}{dt}E(t) + (1-\varpi)\|u_t\|^2 + \varpi(\|\nabla u\|^2 + (u,u_t) + \frac{1}{2}\int_\Omega F(u)dx) \leq c, \tag{51}$$

There exists c_0 such that

$$(1-\varpi)\|u_t\|^2+\varpi(\|\nabla u\|^2+(u,u_t)+\int_\Omega F(u)dx)\geq c_0E(t).$$

We deduce from (50) that

$$\frac{d}{dt}E(t)+c_0E(t)\leq C_0, \tag{52}$$

On the other hand, there exists $c_1, c_2, C, C'\geq 0$ such that

$$c_1(\|\nabla u\|^2+\|u_t\|^2-1)\leq E(t)\leq c_2(\|u_t\|^2+\|u\|_{H^1}^4+1).$$

We apply the Gronwall lemma to (52) and we deduce that

$$E(t)\leq E(0)e^{-c_0t}+C_0(1-e^{-c_0t}).$$

Hence

$$B_1=\{(u,u_t)\in E_1,\ \|\nabla u\|^2+\|u_t\|^2\leq 2\frac{C_0+c_1}{c_1}=\varrho_1^2\}$$

is an absorbing set for $S(t)$ on E_1.

2.) Let

$$(u_0,u_1)\in B=\{(u,v)\in E_1,\ \|(u,v)\|_{E_1}\leq R\},$$

for some $R>0$. We multiply (38) by $-\Delta u$, and we integrate over Ω, and we obtain

$$\frac{d}{dt}(\nabla u_t,\nabla u)+\|\Delta u\|^2+(\nabla u,\nabla u_t)+(\nabla(f(u)+p(u)),\nabla u)=\|\nabla u_t\|^2,$$

and then

$$\frac{d}{dt}(\nabla u_t,\nabla u)+\frac{1}{2}\|\Delta u\|^2+(\nabla u,\nabla u_t)\leq\|\nabla u_t\|^2+\|\nabla(g(u)+p(u))\|^2. \tag{53}$$

We multiply (38) by $-\Delta u_t$, and we integrate over Ω, and we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\Delta u\|^2\right]+\|\nabla u_t\|^2+(\nabla(f(u)+p(u)),\nabla u_t)=0,$$

and then

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\Delta u\|^2\right]+\frac{1}{2}\|\nabla u_t\|^2\leq\|\nabla(f(u)+p(u))\|^2. \tag{54}$$

Summing ϖ(53) and (54), for some small $\varpi\in(0,1)$, we deduce

$$\frac{d}{dt}E(t)+(\frac{1}{2}-\varpi)\|\nabla u_t\|^2+\varpi(\frac{1}{2}\|\Delta u\|^2+(\nabla u,\nabla u_t))\leq c\|\nabla(f(u)+p(u))\|^2, \tag{55}$$

where

$$E(t)=\varpi(\nabla u_t,\nabla u)+\left[\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\Delta u\|^2\right].$$

There exists $t(R) = t_1$ such that

$$\|\nabla u(t)\|^2 + \|u_t(t)\|^2 \le \varrho_1^2, \quad t \ge t_1.$$

From assumption (45), we have that

$$\begin{aligned}\|\nabla(f(u)+p(u))\|^2 \le & c(\varrho_1)(1+\|\Delta u\|)^{2-2\sigma_2} \\ & \le \epsilon(1+\|\Delta u\|)^2 + C_\epsilon(\varrho_1) \\ & \le \epsilon\|\Delta u\|^2 + C_\epsilon(\varrho_1).\end{aligned}$$

for every $\epsilon > 0$. We deduce from (55) that

$$\frac{d}{dt}E(t) + (\frac{1}{2} - \varpi)\|\nabla u_t\|^2 + \varpi(\frac{1}{4}\|\Delta u\|^2 + (u, u_t)) \le c(\varrho_1), \quad \forall t \ge t_1. \tag{56}$$

There exists c_0 such that

$$(\frac{1}{2} - \varpi)\|\nabla u_t\|^2 + \varpi(\frac{1}{4}\|\Delta u\|^2 + (u, u_t)) \ge c_0 E(t).$$

We deduce from (55) that

$$\frac{d}{dt}E(t) + c_0 E(t) \le C_1(\varrho_1), \quad \forall t \ge t_1. \tag{57}$$

On the other hand, there exists $c_1, c_2 \ge 0$ such that

$$c_1(\|\Delta u\|^2 + \|\nabla u_t\|^2) \le E(t) \le c_2(\|u_t\|_{H^1}^2 + \|u\|_{H^2}^2).$$

We apply the Gronwall lemma to (57) and we deduce that

$$E(t) \le E(t_1)e^{-c_0(t-t_1)} + C_1(\varrho_1)(1 - e^{-c_0(t-t_1)}).$$

We can show that $E(t_1) \le c(R)$. Hence

$$B_2 = \{(u, u_t) \in E_1, \|\Delta u\|^2 + \|\nabla u_t\|^2 \le 2\frac{C_1(\varrho_1)}{c_1} = \varrho_2^2\}$$

is an absorbing set for $S(t)$ on E_1.