Problem 1. Let $\Omega$ be an open bounded set of $\mathbb{R}^d$, $d = 1, 2, \text{ or } 3$, with smooth boundary $\partial \Omega$. Consider the initial and boundary value problem (this is the viscous Cahn-Hilliard equation)

\begin{align*}
\frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) &= 0, \quad \text{in } \Omega \times \mathbb{R}_+, \quad (1) \\
u|_{t=0} &= u_0, \quad \text{in } \Omega, \quad (2) \\
\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} &= 0, \quad \text{on } \partial \Omega \times \mathbb{R}_+, \quad (3)
\end{align*}

where $n$ is the unit outward normal on $\partial \Omega$, $u(x, t), x \in \Omega, t \in \mathbb{R}_+$, is the unknown scalar function, $f \in C^1(\mathbb{R})$ is a function satisfying the conditions:

\begin{align*}
|f'(s)| &\leq c_1(|s|^2 + 1), \quad \forall s \in \mathbb{R}, \quad (4) \\
sf(s) &\geq F(s) - c_2, \quad \forall s \in \mathbb{R}, \quad (5) \\
f'(s) &\geq -c_3, \quad \forall s \in \mathbb{R}, \quad (6)
\end{align*}

for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $|f(s)| \leq \epsilon F(s) - C_\epsilon, \quad \forall s \in \mathbb{R}$, \quad (7)

\begin{align*}
F(s) &\geq -c_4, \quad \forall s \in \mathbb{R}, \quad (8)
\end{align*}

where $F(s) = \int_0^s f(\tau)d\tau$.

We introduce the space

$$V = \{u \in H^2(\Omega), \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega\}.$$

1.) Implement Galerkin approximation method to prove that, if $u_0 \in H^1(\Omega)$, then problem (1)-(3) possesses a unique solution $u$ which satisfies

$$u \in C([0, T]; H^1(\Omega)) \cap L^2(0, T; V), \quad u_t \in L^2(0, T; L^2(\Omega)), \quad \forall T > 0.$$

2.) Prove that, if $u_0 \in V$, then

$$u \in L^\infty(0, T; V), \quad u_t \in L^2(0, T; H^1(\Omega)), \quad \forall T > 0.$$

Solution: If $X$ is a Sobolev-type space, then we set

$$\dot{X} = \{q \in X, m(q) = 0\},$$

where

$$m(q) = \frac{1}{L} \int_\Omega q(x) \, dx.$$

Moreover, we set

$$\bar{q} = q - m(q).$$
Let us define the linear unbounded operator

$$-\Delta : V \to \dot{H}$$

which is self-adjoint and nonnegative. If $-\Delta$ is restricted to $\dot{V}$, then it turns to be positive with compact inverse $(-\Delta)^{-1}$. There is a complete orthonormal family $w_j$ on $L^2(\Omega)$ made of eigenvectors of $-\Delta$ such that

$$-\Delta w_j = \lambda_j w_j, \quad w_j \in V,$$

(in fact, $w_j \in W$, since $\partial \Omega$ is sufficiently regular) and

$$0 = \lambda_1 < \lambda_2 < \ldots < \lambda_m < \ldots$$

For every $r > 0$, we endow $(D(\Delta)^{r/2})'$ with the norm $\|q\|_r = (\|(-\Delta)^{-r/2}q\|^2 + |m(q)|^2)^{1/2}$. We also note that $\|q\|_r = (\|(-\Delta)^{r/2}q\|^2 + |m(q)|^2)^{1/2}$ is a norm on $D((-\Delta)^{r/2})$ which is equivalent to the usual $H^r(\Omega)$-norm.

1.)a.) Weak formulation: For any given $T > 0$, find $u : [0, T] \to V$ such that

$$u(0) = u_0,$$

and, for almost every $t \in [0, T]$,

$$\frac{d}{dt}[(u, q) + (\nabla u, \nabla q)] + (\Delta u, \Delta q) - (f'(u)\nabla u, \nabla q) = 0, \quad \forall q \in V. \quad (9)$$

b.) Let take $q = 1$ (which belongs to $V$) in (9). We deduce that

$$\frac{d}{dt} \int_{\Omega} u(x, t)dx = 0. \quad (10)$$

We integrate (10) between 0 and $t$, and we find

$$\int_{\Omega} u(x, t)dx = \int_{\Omega} u(x, 0)dx, \quad \forall t \geq 0. \quad (11)$$

c.) We set $E_m = \text{span}\{w_1, \ldots, w_m\}$ and $P_m$ is the orthogonal projection on $E_m$. We consider the approximate problem:

For each $m$, find an approximate solution $u_m$ of the form $u_m(t) = \sum_{j=1}^{m} c_j(t)w_j$ satisfying

$$\frac{d}{dt}[(u_m, w_j) + (\nabla u_m, \nabla w_j)] + (\Delta u_m, \Delta w_j) + (f'(u_m)\nabla u_m, \nabla w_j) = 0, \quad j = 1, \ldots, m, \quad (12)$$

$$u_m(0) = P_m u_0, \quad (13)$$

d.) Problem (12)-(13) yields

$$(1 + \lambda_j)c_j' + c_j \lambda_j^2 + (f'(u_m)\nabla u_m, \nabla w_j) = 0, \quad j = 1, \ldots, m. \quad (14)$$

Now, setting $Y = (c_1, \ldots, c_m)$ we infer from (14) a system of the form:

$$M_1 Y' + M_2 Y + F(Y) = 0, \quad (15)$$
where
\[ M_1 = ((1 + \lambda_i)\delta_{ij})_{i,j=1,...,m}, \quad M_2 = (\lambda_i^2\delta_{ij})_{i,j=1,...,m} \]
and
\[ F(Y) = ((f'(\sum_{j=1}^{m} c_j w_j) \sum_{j=1}^{m} c_j \nabla w_j, \nabla w_1), \ldots, (f'(\sum_{j=1}^{m} c_j w_j) \sum_{j=1}^{m} c_j \nabla w_j, \nabla w_m)). \]

The matrix \( M_1 \) is invertible and \( F(Y) \) depends continuously on \( Y \). Hence, (12)-(13) has a unique solution on some finite time interval \([0, T_m] \).

Now, we have to show that the solutions are bounded in time and uniformly bounded in \( m \). First we note that (12) is also equivalent to

\[ \frac{du_m}{dt} - \Delta u_m + \Delta^2 u_m - P_m \Delta f(u_m) = 0, \quad (16) \]

We multiply (16) by \( u_m \), integrate over \( \Omega \) and we find

\[ \frac{1}{2} \frac{d}{dt}[\|u_m\|^2 + \|
abla u_m\|^2] + \|\Delta u_m\|^2 = \int_{\Omega} f'(u_m)|\nabla u_m|^2 dx = 0. \]

Using (6), we deduce

\[ \frac{1}{2} \frac{d}{dt}[\|u_m\|^2 + \|
abla u_m\|^2] + \|\Delta u_m\|^2 \leq c_{10} \|
abla u_m\|^2, \quad (17) \]

Neglecting the two terms in the middle, we find

\[ \frac{d}{dt}[\|u_m\|^2 + \|
abla u_m\|^2] \leq c(\|u_m\|^2 + \|
abla u_m\|^2), \quad (18) \]

We apply the Gronwall lemma to (18), and noting that \( \|u_m(0)\|^2 + \|
abla u_m(0)(t)\|^2 \leq \|u_0\|^2 + \|
abla u_0\|^2 \), we get

\[ \|u_m(t)\|^2 + \|
abla u_m\|^2 \leq c[\|u_0\|^2 + \|
abla u_0\|^2]e^{ct}, \quad \forall t \geq 0, \]

hence

\[ \sup_{t \in [0,T]} \|u_m(t)\|^2_{H^1} \leq C_1. \quad (19) \]

Now, we add \( m(u_m)^2 \) in both sides of (17), and we integrate the resulting equation between 0 and \( T \), and we find

\[ \|u_m(T)\|^2 + \|
abla u_m(T)\|^2 + \int_0^T (m(u_m))^2 + \|\Delta u_m\|^2)dt \leq c \int_0^T (m(u_0))^2 + \|
abla u_m\|^2)dt \leq C(T), \]

It follows that

\[ \int_0^T \|u_m\|^2_{H^2}dt \leq C(T). \quad (20) \]
Lastly, we deduce from (16) that
\[
\frac{du_m}{dt} = -(I - \Delta)^{-1} \Delta [\Delta u_m - f(u_m)].
\] (21)

We have that
\[
\Delta u_m \in L^2(0, T; L^2(\Omega)).
\]

We have
\[
\int_0^T \|f(u_m)\|^2 dt \leq \int_0^T \int_\Omega |f(u_m)|^2 dx dt \\
\leq c_3 \int_0^T \int_\Omega u_m^6 dx + c_4 T \\
\leq c_3 \int_0^T \|u_m\|^6_{L^6(\Omega)} dt + c_4 T \\
\leq c_3 \int_0^T \|u_m\|^6_{H^1(\Omega)} dt + c_4 T \\
\leq C(T),
\] (22)
on account of (19). From the regularity of elliptic problems, we know that
\[
\|(I - \Delta)^{-1} u\|_{H^2} \leq c\|u\|.
\]

Therefore
\[
\|(I - \Delta)^{-1} \Delta u\| \leq \|(I - \Delta)^{-1} u\|_{H^2} \leq c\|u\|
\]

We can deduce from (21) that
\[
\|\frac{du_m}{dt}\| \leq c\|\Delta u_m - f(u_m)\|,
\] (23)

and therefore
\[
\int_0^T \left\|\frac{du_m}{dt}\right\|^2 dt \leq c \int_0^T \|\Delta u_m\|^2 dt + \int_0^T \|f(u_m)\|^2 dt \\
\leq C(T).
\] (24)

We could also have proceeded as follows. Since \(u_m \in L^2(0, T; V)\), we have that \(\Delta^2 u_m \in L^2(0, T; V')\). On the other hand, since \(f(u_m) \in L^2(0, T; L^2(\Omega))\), it follows that \(-\Delta f(u_m) \in L^2(0, T; V')\). We deduce from (16) that \((I - \Delta)\frac{du_m}{dt} = -\Delta^2 u_m + \Delta f(u)\), and consequently that \((I - \Delta)\frac{du_m}{dt} \in L^2(0, T; V')\), hence \(\frac{du_m}{dt} \in L^2(0, T; L^2(\Omega))\).

The constants from \(C_1\) through \(C_4\) are all bounded for bounded sets of initial conditions and bounded time intervals. These estimates are also bounded uniformly in \(m\). This yields that the solution is global (i.e. \(T_m = T\)). We have the existence of a convergent subsequence in the respective Sobolev spaces and we can pass to the limit. The fact that \(u \in L^2(0, T; V)\) and \(\frac{du}{dt} \in L^2(0, T; L^2(\Omega))\) implies that \(u \in C([0, T]; L^2(\Omega))\). The fact that \(u \in L^\infty(0, T, H^1(\Omega))\) implies that \(u \in C_w([0, T]; H^1(\Omega))\). The fact that the map \([0, T] \to H^1(\Omega), t \mapsto \|\nabla u(t)\| is
continuous implies that \( u \in C([0, T]; H^1(\Omega)) \).

Indeed, we observe that, for \( u \in V \) and \( g \in L^2(\Omega) \), we have

\[
-\Delta u = \bar{g} \iff u = (-\Delta)^{-1} \bar{g}.
\]

Thus, the equation (16) can be rewritten as

\[
-\Delta \left[ \frac{d}{dt} (-\Delta)^{-1} \bar{u}_m + \frac{d u_m}{dt} - \Delta u_m + P_m f(u_m) \right] = 0,
\]

which is equivalent to

\[
\frac{d}{dt} (-\Delta)^{-1} \bar{u}_m + u_m - \Delta u_m + P_m f(u_m) = 0.
\] (25)

(since \((-\Delta)^{-1} 0 = 0\)).

We multiply (25) by \( \frac{d u_m}{dt} \), integrate over \( \Omega \) and we find

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \nabla u_m \|^2 + \| \frac{d u_m}{dt} \|^2 + \| \frac{d u_m}{dt} \|^{-1}_2 + \int_{\Omega} (f(u_m), \frac{d u_m}{dt}) \right] dx = 0.
\]

We integrate between \( t_0 \) and \( t \), and we find

\[
\| \nabla u_m(t) \|^2 + 2 \int_{t_0}^t \| \frac{d u_m}{dt} \|^2 dx + 2 \int_{t_0}^t \| \frac{d u_m}{dt} \|^{-1}_2 dx + 2 \int_{t_0}^t (f(u_m), \frac{d u_m}{dt}) dx = \| \nabla u_m(t_0) \|^2.
\]

All the integral terms tend to zero when \( t \) tends to \( t_0 \), hence

\[
\lim_{t \to t_0} \| \nabla u_m(t) \|^2 = \| \nabla u_m(t_0) \|^2.
\]

Uniqueness: Let \( w = u_1 - u_2 \), where \( u_1 \) and \( u_2 \) are two solutions of (1)-(3). There holds \( w(0) = 0 \) and \( m(w) = 0 \). We multiply (25) by \( w \), and we integrate over \( \Omega \), and we find

\[
\frac{1}{2} \frac{d}{dt} \left[ \| w \|^2 + \| \frac{d u_m}{dt} \|^2 + \| \frac{d u_m}{dt} \|^2_1 + \int_{\Omega} (f(u_1) - f(u_2)) w dx = 0. \right]
\] (26)

We have \( f(u_1) - f(u_2) = f'(c)w \), with \( c = \theta u_1 + (1 - \theta) u_2, \theta \in (0, 1) \); hence

\[
\int_{\Omega} (f(u_1) - f(u_2)) w dx = \int_{\Omega} f'(c)|w|^2 dx 
\geq -C \| w \|^2.
\]

We deduce from (26) that

\[
\frac{1}{2} \frac{d}{dt} \left[ \| w \|^2 + \| \frac{d u_m}{dt} \|^2_1 + \| \frac{d u_m}{dt} \|^2 \right] \leq C \| w \|^2,
\]

and then

\[
\frac{d}{dt} [\| w \|^2 + \| \frac{d u_m}{dt} \|^2_1 + \| \frac{d u_m}{dt} \|^2] \leq c(\| w \|^2 + \| \frac{d u_m}{dt} \|^2).
\] (27)
We multiply by $e^{-ct}$ and integrate between 0 and $t$, and we find
$$\|w(t)\|_{L^2}^2 + \|w(t)\|^2 \leq e^{ct}(\|w(0)\|_{L^2}^2 + \|w(0)\|^2), \quad \forall t \geq 0;$$
hence $w(t) = 0$, and $u_1(t) = u_2(t)$.

2.) We multiply (16) by $\frac{d u_m}{d t}$, integrate over $\Omega$ and we find
$$\frac{1}{2} \frac{d}{d t} \|\Delta u_m\|^2 + \|\frac{d u_m}{d t}\|^2 + \|\nabla \frac{d u_m}{d t}\|^2 = \int_{\Omega} \nabla f(u_m) \nabla \frac{d u_m}{d t} \, dx = 0.$$  
We have
$$\left| \int_{\Omega} \nabla f(u_m) \nabla \frac{d u_m}{d t} \, dx \right| \leq \int_{\Omega} \|\nabla f(u_m)\| \|\nabla u_m\| \|\nabla \frac{d u_m}{d t}\| \, dx$$
$$\leq \|f'(u_m)\| L^6 \|\nabla u_m\| L^6 \|\nabla \frac{d u_m}{d t}\|$$
$$\leq c(\|u_m\|_{L^6}^2 + 1) \|\nabla u_m\|_{L^6} \|\nabla \frac{d u_m}{d t}\|$$
$$\leq c(\|u_m\|_{H^1}^2 + 1) \|u_m\|_{H^2} \|\nabla \frac{d u_m}{d t}\|.$$  
We then get and then
$$\frac{d}{d t} \|\Delta u_m\|^2 + \|\frac{d u_m}{d t}\|^2 \leq c(\|u_m\|_{H^1}^2 + 1)^2 \|u_m\|_{H^2}^2. \quad (28)$$
Noting that
$$\|u_m\|_{H^2}^2 \leq c(|m(u_m)|^2 + \|\Delta u_m\|),$$
we can deduce from (28) that
$$\sup_{t \in [0, T]} \|u\|_{H^2}^2 \leq C(T), \quad (29)$$
and
$$\int_0^T \|\frac{d u_m}{d t}\|_{H^1}^2 \, dt \leq C(T). \quad (30)$$

The constants from $C_5$ through $C_7$ are all bounded for bounded sets of initial conditions and bounded time intervals. These estimates are also bounded uniformly in $m$. We have the existence of a convergent subsequence in the respective Sobolev spaces and we can pass to the limit.
Problem 2. We introduce the spaces
\[ H_\beta = \left\{ q \in H^1(\Omega), \frac{1}{|\Omega|} \int_\Omega q(x)dx = \beta \right\}, \quad \beta \in \mathbb{R}, \]
\[ \mathcal{H}_\alpha = \bigcup_{|\beta| \leq \alpha} H_\beta. \]

We define the semigroup
\[ S(t) : H^1(\Omega) \to H^1(\Omega), \quad u_0 \mapsto u(t), \]
where \( u(t) \) is the solution of (1)-(3) (see Problem 1) at time \( t \).

1.) Show that \( S(t) \) has an absorbing set \( B_1 \) in \( \mathcal{H}_\alpha \).
3.) Show that \( S(t) \) has an absorbing set \( B_2 \) in \( \mathcal{H}_\alpha \cap H^2(\Omega). \)
4.) Deduce that, for every \( \alpha \geq 0 \), \( S(t) \) has a global attractor \( A_\alpha \) in \( \mathcal{H}_\alpha \).

Solution:
1.) let \( u_0 \in B = \{ u \in H^1(\Omega), \| u \|_{H^1} \leq R, \| m(u) \| \leq \alpha \} \) for some \( R > 0 \). We saw that (1) can be reduced to
\[ \frac{d}{dt} \left[ (-\Delta)^{-1} \bar{u} + u \right] - \Delta u + f(u) = 0. \quad (31) \]

Now, we multiply (31) by \( \bar{u} \) and integrate over \( \Omega \), and we find
\[ \frac{1}{2} \frac{d}{dt} \left[ \| \bar{u} \|_{-1}^2 + \| \bar{u} \|^2 + \| \nabla u \|^2 \right] + \int_\Omega f(u)u dx - m(u) \int_\Omega f(u) dx = 0. \]

Using Assumption (5), we deduce that
\[ \frac{1}{2} \frac{d}{dt} \left[ \| \bar{u} \|_{-1}^2 + \| \bar{u} \|^2 + \| \nabla u \|^2 \right] + c \int_\Omega F(u) dx \leq |m(u_0)| \int_\Omega f(u) dx + c_2|\Omega| \]
\[ \leq \alpha \int_\Omega |f(u)| dx + c_2|\Omega|. \]

Using now assumption (7), we deduce that
\[ \frac{1}{2} \frac{d}{dt} \left[ \| \bar{u}_m \|_{-1}^2 + \| \bar{u}_m \|^2 + \| \nabla u_m \|^2 \right] + c \int_\Omega F(u) dx \leq c_1(\alpha). \quad (32) \]

We multiply (1) by \( \frac{du}{dt} \), integrate over \( \Omega \) and we find
\[ \frac{1}{2} \frac{d}{dt} \left[ \| \nabla u \|^2 + 2 \int_\Omega F(u) dx \right] + \| \frac{du}{dt} \|_{-1}^2 + \| \frac{du}{dt} \|^2 = 0. \quad (33) \]

Summing (32) and (33), we get
\[ \frac{1}{2} \frac{d}{dt} E_1(t) + \| \frac{du}{dt} \|^2 + \| \frac{du}{dt} \|_{-1}^2 + \| \nabla u_m \|^2 \right] + c \int_\Omega F(u) dx \leq c_1(\alpha). \quad (34)
where
\[ E_1(t) = \|\bar{u}_m\|_1^2 + \|\bar{u}_m\|^2 + \|\nabla u\|^2 + 2 \int_{\Omega} F(u) dx \]

We get from the Poincaré inequality that
\[ \|\bar{u}\| \leq c\|\nabla u\|, \quad \forall u \in H^1(\Omega). \]

Since \( L^2(\Omega) \subset H^{-1}(\Omega) \), we also have
\[ \|\bar{u}\|_1^{-1} \leq c\|\bar{u}\|, \quad \forall u \in L^2(\Omega). \]

There exists \( c_0 > 0 \) such that
\[ \|\nabla u_m\|^2 + c \int_{\Omega} F(u) dx \geq c_0 E_1(t) \]

We then get from (34) that
\[ \frac{d}{dt} E_1(t) + c_0 E_1(t) \leq c_3(\alpha). \] (35)

Applying the Gronwall’s lemma, we find
\[ E_1(t) \leq E_1(0)e^{-c_0 t} + c_3(\alpha)(1 - e^{-c_0 t}), \quad \forall t \geq 0. \]

We now note that there exist \( c_1, c_2 > 0 \) such that
\[ \|\nabla u\|^2 - c_1 \leq E_1(t) \leq c_2(\|u\|^4_{H^1} + 1). \]

Thus,
\[ B_1 = \{u \in H^1(\Omega), \ |m(u)|^2 + \|\nabla u\|^2 \leq 2c_3(\alpha) + c_1 + \alpha^2 = \rho_1^2\} \]
is an absorbing set for \( S(t) \).

2.) We recall from (28) Problem 1 that
\[ \frac{d}{dt}\|\Delta u\|^2 \leq c(\|u\|^2_{H^1} + 1)^2\|u\|^2_{H^2} \] (36)

There exists \( t(R) > 0 \) such that
\[ |m(u(t))|^2 + \|\nabla u(t)\|^2 \leq \rho_1^2, \quad \forall t \geq t(R) = t_1. \]

We integrate (35) between \( t \) and \( t + 1 \), for any \( t \geq t_1 \), we deduce
\[ c_0 \int_t^{t+1} E_1(s) ds \leq E_1(t) + c_3(\alpha), \quad \forall t \geq t_1, \]

hence
\[ \int_t^{t+1} \|u\|^2_{H^1} ds \leq c_4(\rho_1), \quad \forall t \geq t_1. \]
Now, we integrate (17) between $t_1$ and $t$, we deduce

$$\int_{t}^{t+1} \|u\|_{H^2}^2 ds \leq c_5(\varrho_1), \ \forall t \geq t_1.$$  

We deduce from (36) that

$$\frac{d}{dt} \|\Delta u\|^2 \leq c_6(\varrho_1)\|\Delta u\|^2 + c_7(\varrho_1), \ \forall t \geq t_1$$

(37)

WE now apply the uniform Gronwall lemma to (37), and we find that

$$\|\Delta u(t)\|^2 \leq c_8(\varrho_1), \ \forall t \geq t_1 + 1$$

Thus,

$$B_2 = \{u \in H^2(\Omega), \ |m(u)|^2 + \|\Delta u\|^2 \leq c_8(\varrho_1) + \alpha^2 = \varrho_2^2\}$$

is an absorbing set for $S(t)$.

3.) The semigroup $S(t)$ is uniformly compact and has a global attractor $A$ in $\mathcal{H}_\alpha$.
**Problem 3.** Let \( \Omega \) be an open bounded set of \( \mathbb{R}^d, \ d \leq 3, \) with smooth boundary \( \partial \Omega. \) Consider the initial and boundary value problem

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) + p(u) = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{38}
\]

\[
u(x, 0) = u_0, \quad \frac{\partial u}{\partial t}(x, 0) = u_1, \quad \text{in } \Omega, \tag{39}
\]

\[
u = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+, \tag{40}
\]

where \( u(x, t), \ x \in \Omega, \ t \in \mathbb{R}_+, \) is the unknown scalar function. We assume that \( f, p \in \mathcal{C}^1(\mathbb{R}) \) and

\[
F(s) \geq -c_1, \quad \forall s \in \mathbb{R}, \tag{41}
\]

\[
sf(s) - c_2 F(s) \geq -c_3, \quad \forall s \in \mathbb{R}, \tag{42}
\]

\[
|f'(s)| \leq c_4 (|s|^2 + 1), \quad \forall s \in \mathbb{R}, \tag{43}
\]

\[
|p(s)| \leq c_5 (F(s) + 1)^{1 - \sigma_1}, \quad \sigma_1 > 0, \tag{44}
\]

and

\[
\text{if } \|\nabla u\| \leq R, \text{ then there exists } C(R) > 0 \text{ such that } \|\nabla (f(u) + p(s))\| \leq C(R)(1 + \|\Delta u\|)^{1 - \sigma_2}, \quad \sigma_2 > 0, \tag{45}
\]

where \( F(s) = \int_0^s f(\zeta) d\zeta. \)

Let introduce the spaces

\( E_1 = H^1_0(\Omega) \times L^2(\Omega), \quad E_2 = H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega). \)

We can define the semigroup

\[
S(t): \ E_j \to E_j, \quad j = 1, 2, \quad (u_0, u_1) \mapsto (u(t), u_t(t)),
\]

where \( u(t) \) is the solution of (38)-(40) at time \( t. \)

1.) Show that \( S(t) \) has an absorbing set \( B_1 \) in \( E_1. \)

2.) Show that \( S(t) \) has an absorbing set \( B_2 \) in \( E_2. \)

**Solution:**

1.) Let

\[
(u_0, u_1) \in B = \{(u, v) \in E_1, \quad \|(u, v)\|_{E_1} \leq R\},
\]

for some \( R > 0. \) We multiply (38) by \( u, \) and we integrate over \( \Omega, \) and we obtain

\[
\frac{d}{dt} (u_t, u) + \|\nabla u\|^2 + (u, u_t) + (f(u), u) + (p(u), u) = \|u_t\|^2. \tag{46}
\]
Using assumption (42), we have
\[ \int_{\Omega} f(u) \, dx \geq c_1 \int_{\Omega} F(u) \, dx - C. \]

We infer from (46) that
\[ \frac{d}{dt}(u_t, u) + \|\nabla u\|^2 + (u, u_t) + c_1 \int_{\Omega} F(u) \, dx \leq \|u_t\|^2 + \|u\|\|p(u)\| \]
\[ \leq \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + c\|p(u)\|^2, \]
hence
\[ \frac{d}{dt}(u_t, u) + \frac{1}{2} \|\nabla u\|^2 + (u, u_t) + c_1 \int_{\Omega} F(u) \, dx \leq \|u_t\|^2 + c\|p(u)\|^2. \tag{47} \]

We multiply (38) by \( u_t \), and we integrate over \( \Omega \), and we obtain
\[ \frac{d}{dt} \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \int_{\Omega} F(u) \, dx \right] + \|u_t\|^2 + (p(u), u_t) = 0. \tag{48} \]

We then deduce that
\[ \frac{d}{dt} \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \int_{\Omega} F(u) \, dx \right] + \|u_t\|^2 \leq \|p(u)\|\|u_t\| \]
\[ \leq \frac{1}{2} \|u_t\|^2 + c\|p(u)\|^2, \]
hence
\[ \frac{d}{dt} \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \int_{\Omega} F(u) \, dx \right] + \|u_t\|^2 \leq c\|p(u)\|^2, \tag{49} \]

Summing \( \varpi \)(47) and (49), for some small \( \varpi \in (0, 1) \), we deduce
\[ \frac{d}{dt} E(t) + (1 - \varpi)\|u_t\|^2 + \varpi(\|\nabla u\|^2 + (u, u_t) + \int_{\Omega} F(u) \, dx) \leq c\|p(u)\|^2, \tag{50} \]
where
\[ E(t) = \varpi(u_t, u) + \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \int_{\Omega} F(u) \, dx \right]. \]

Finally, we have
\[ \|p(u)\|^2 \leq c_2 \int_{\Omega} (F(s) + 1)^{1-2q_2} \, dx \]
\[ \leq \epsilon \int_{\Omega} F(s) \, dx + c_\epsilon, \quad \forall \epsilon > 0. \]

We deduce from (50) that
\[ \frac{d}{dt} E(t) + (1 - \varpi)\|u_t\|^2 + \varpi(\|\nabla u\|^2 + (u, u_t) + \frac{1}{2} \int_{\Omega} F(u) \, dx) \leq c, \tag{51} \]
There exists $c_0$ such that

$$(1 - \varpi)\|u_t\|^2 + \varpi(\|\nabla u\|^2 + (u, u_t) + \int_{\Omega} F(u) dx) \geq c_0 E(t).$$

We deduce from (50) that

$$\frac{d}{dt} E(t) + c_0 E(t) \leq C_0,$$  \hspace{1cm} (52)

On the other hand, there exists $c_1, c_2, C, C' \geq 0$ such that

$$c_1(\|\nabla u\|^2 + \|u_t\|^2 - 1) \leq E(t) \leq c_2(\|u_t\|^2 + \|u\|_{H^1}^4 + 1).$$

We apply the Gronwall lemma to (52) and we deduce that

$$E(t) \leq E(0)e^{c_0 t}.$$ 

Hence

$$B_1 = \{(u, u_t) \in E_1, \|\nabla u\|^2 + \|u_t\|^2 \leq 2\frac{C_0 + c_1}{c_1} = \varrho_1^2\}$$

is an absorbing set for $S(t)$ on $E_1$.

2.) Let

$$(u_0, u_1) \in B = \{(u, v) \in E_1, \|(u, v)\|_{E_1} \leq R\},$$

for some $R > 0$. We multiply (38) by $-\Delta u$, and we integrate over $\Omega$, and we obtain

$$\frac{d}{dt}(\nabla u_t, \nabla u) + \|\Delta u\|^2 + (\nabla u, \nabla u_t) + (\nabla(f(u) + p(u)), \nabla u) = \|\nabla u_t\|^2,$$

and then

$$\frac{d}{dt}(\nabla u_t, \nabla u) + \frac{1}{2}\|\Delta u\|^2 + (\nabla u, \nabla u_t) \leq \|\nabla u_t\|^2 + \|\nabla(g(u) + p(u))\|^2.$$  \hspace{1cm} (53)

We multiply (38) by $-\Delta u_t$, and we integrate over $\Omega$, and we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2\right] + \|\nabla u_t\|^2 + (\nabla(f(u) + p(u)), \nabla u_t) = 0,$$

and then

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2\right] + \frac{1}{2}\|\nabla u_t\|^2 \leq \|\nabla(f(u) + p(u))\|^2.$$  \hspace{1cm} (54)

Summing $\varpi(53)$ and (54), for some small $\varpi \in (0, 1)$, we deduce

$$\frac{d}{dt} E(t) + \frac{1}{2} - \varpi\|\nabla u_t\|^2 + \varpi(\frac{1}{2}\|\Delta u\|^2 + (\nabla u, \nabla u_t)) \leq c\|\nabla(f(u) + p(u))\|^2,$$  \hspace{1cm} (55)

where

$$E(t) = \varpi(\nabla u_t, \nabla u) + \left[\frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2\right].$$
There exists \( t(R) = t_1 \) such that
\[
\|\nabla u(t)\|^2 + \|u_t(t)\|^2 \leq \varphi_1^2, \quad t \geq t_1.
\]

From assumption (45), we have that
\[
\|\nabla (f(u) + p(u))\|^2 \leq c(\varphi_1)(1 + \|\Delta u\|)^{2-2\sigma_2}
\leq \epsilon(1 + \|\Delta u\|)^2 + C\epsilon(\varphi_1)
\leq \epsilon\|\Delta u\|^2 + C\epsilon(\varphi_1).
\]
for every \( \epsilon > 0 \). We deduce from (55) that
\[
\frac{d}{dt}E(t) + \frac{1}{2}(1 - \varpi)\|\nabla u_t\|^2 + \varpi(\frac{1}{4}\|\Delta u\|^2 + (u, u_t)) \leq c(\varphi_1), \quad \forall t \geq t_1. \tag{56}
\]

There exists \( c_0 \) such that
\[
(\frac{1}{2} - \varpi)\|\nabla u_t\|^2 + \varpi(\frac{1}{4}\|\Delta u\|^2 + (u, u_t)) \geq c_0E(t).
\]

We deduce from (55) that
\[
\frac{d}{dt}E(t) + c_0E(t) \leq C_1(\varphi_1), \quad \forall t \geq t_1. \tag{57}
\]

On the other hand, there exists \( c_1, c_2 \geq 0 \) such that
\[
c_1(\|\Delta u\|^2 + \|\nabla u_t\|^2) \leq E(t) \leq c_2(\|u_t\|_{H^1}^2 + \|u\|_{H^2}^2).
\]

We apply the Gronwall lemma to (57) and we deduce that
\[
E(t) \leq E(t_1)e^{-c_0(t-t_1)} + C_1(\varphi_1)(1 - e^{-c_0(t-t_1)}).
\]

We can show that \( E(t_1) \leq c(R) \). Hence
\[
B_2 = \{(u, u_t) \in E_1, \|\Delta u\|^2 + \|\nabla u_t\|^2 \leq 2\frac{C_1(\varphi_1)}{c_1} = \varphi_2^2\}
\]
is an absorbing set for \( S(t) \) on \( E_1 \).