MATH 101 - Exam I - Term 141
Duration: 90 minutes

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 6 Problems)

<table>
<thead>
<tr>
<th>Page Number</th>
<th>Points</th>
<th>Maximum Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>
1. (7 points) Sketch the graph of a function $f$ that satisfies the following conditions:

(i) $f(0) = 0$,

(ii) $\lim_{x \to -\infty} f(x) = 1$,

(iii) $f$ has a jump discontinuity at $x = -1$,

(iv) $\lim_{x \to 0^-} f(x) = -\infty$,

(v) $\lim_{x \to 0^+} f(x) = \infty$,

(vi) $f$ has a removable discontinuity at $x = 2$.

(vii) $\lim_{x \to \infty} f(x) = 0$

1 point for each condition.

Other graphs are possible.
2. Find the limit if it exists. Justify your work.

a) (3 points) \( \lim_{{x \to 0}} \frac{9 - x}{3 - \sqrt{x}} \)

\[
\lim_{{x \to 0}} \frac{9 - \sqrt{x}}{3 - \sqrt{x}} = \lim_{{x \to 0}} \frac{(3 - \sqrt{x})(3 + \sqrt{x})}{(3 - \sqrt{x})} = \lim_{{x \to 0}} (3 + \sqrt{x}) = 6
\]

(2 pts)

(1 pt)

b) (3 points) \( \lim_{{z \to -\infty}} \frac{x}{{\sqrt{z^2 + 1}}} \)

\[
\lim_{{z \to -\infty}} \frac{x}{{\sqrt{z^2 + 1}}} = \lim_{{z \to -\infty}} \frac{x}{{\sqrt{z^2 (1 + \frac{1}{z^2})}}} = \lim_{{z \to -\infty}} \frac{x}{{\sqrt{1 + \frac{1}{z^2}}}} = \lim_{{z \to -\infty}} -x \sqrt{1 + \frac{1}{z^2}} = \lim_{{z \to -\infty}} -x \]

(1 pt)

(1 pt)

(1 pt)

\[
\lim_{{z \to -\infty}} \frac{1}{{\sqrt{1 + \frac{1}{z^2}}}} = \frac{1}{-1} = -1
\]

(1 pt)
3. (4 points) \( \lim_{x \to 0} \sin^2 x \cos \left( \frac{1}{x} \right) \)

\[-1 \leq \cos \left( \frac{1}{x} \right) \leq 1, \quad x \neq 0\]

\[-\sin^2 x \leq \sin^2 x \cos \left( \frac{1}{x} \right) \leq \sin^2 x\] (1 pt)

Since \( \lim_{x \to 0} -\sin^2 x = 0 = \lim_{x \to 0} \sin^2 x \),

or Sandwich

then by the Squeeze Theorem (1 pt)

\[\lim_{x \to 0} \sin^2 x \cos \left( \frac{1}{x} \right) = 0\] (1 pt)

4. (4 points) \( \lim_{x \to 0} \frac{\sin(3x) \cot(5x)}{x \cot(4x)} \)

\[= \lim_{x \to 0} \frac{\sin(3x)}{x} \cdot \frac{\cos(4x)}{\sin(5x)}\] (1 pt)

\[= \lim_{x \to 0} \left( \frac{\sin(3x)}{3x} \right) \cdot \frac{\cos(4x)}{\sin(5x)}\] (1 pt)

\[= \frac{3}{4} \cdot \frac{\cos(4x)}{\sin(5x)}\] (1 pt)

\[= \left( 3 \right) \left( \frac{4}{5} \right) \left( 1 \right) = \frac{12}{5}\] (1 pt)
3. (7 points) Find the values of \( a \) and \( b \) so that the following function is continuous everywhere.

\[
f(x) = \begin{cases} 
  x^2 - 4x - 2 & \text{if } x < 2 \\
  9x^2 - bx + 4 & \text{if } 2 \leq x < 3 \\
  2x - a + b & \text{if } x \geq 3 
\end{cases}
\]

(i) \( f \) is continuous for \( x < 2 \) (as it is a polynomial)

(ii) for \( f \) to be continuous at \( x = 2 \), we must have

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \quad (1 \text{ pt})
\]

\[
f(2) = 40 - 2b
\]

\[
\lim_{x \to 2^-} x^2 - 4x - 2 = -6 \quad \text{and} \quad \lim_{x \to 2^+} (9x^2 - bx + 4) = 40 - 2b \quad (1 \text{ pt})
\]

\[
\Rightarrow 40 - 2b = -6 \quad \Rightarrow 2b = 46 \quad \Rightarrow b = 23 \quad (1 \text{ pt})
\]

(iii) at \( x = 3 \)

\[
\lim_{x \to 3^+} (2x - a + b) = \lim_{x \to 3^-} (9x^2 - bx + 4) = f(3) \quad (1 \text{ pt})
\]

\[
\Rightarrow 6 - a + b = 85 - 3b \quad (1 \text{ pt})
\]

\[
\Rightarrow 6 - a + 23 = 85 - 69
\]

\[
\Rightarrow a = 13 \quad (1 \text{ pt})
\]
a) (4 points) Use the Intermediate Value Theorem to show that the equation $2 - e^x = \sqrt{x}$ has a root between 0 and 1.

1. $f(x) = 2 - e^x - \sqrt{x}$ (1 pt)
   i) $f$ is **continuous** on $[0,1]$ (1 pt)
   ii) $f(0) = 2 - 1 = 1 > 0$ and $f(1) = 2 - e - 1 = 1 - e \leq 0$

   Then, by Intermediate Value Theorem there is $0 < c < 1$ such that $f(c) = 0$. i.e. $2 - e - \sqrt{c} = 0$ or $2 - e = \sqrt{c}$.

b) (4 points) Use the graph of $f(x) = 2\sqrt{x+1}$ to find a number $\delta > 0$ such that for all $x$,

$$0 < |x - 3| < \delta \Rightarrow |f(x) - 4| < 0.2$$

$a = 3$, $L = 4$, $\varepsilon = 0.2$

$f(x_0) = 3.8$

$\Rightarrow 2\sqrt{x_0 + 1} = 3.8 \Rightarrow \sqrt{x_0 + 1} = 1.9$

$\Rightarrow x_0 + 1 = 3.61 \Rightarrow x_0 = 2.61$ (1 pt)

$f(x_0) = 4.2$

$\Rightarrow 2\sqrt{x_1 + 1} = 4.2$

$\Rightarrow \sqrt{x_1 + 1} = 2.1 \Rightarrow x_1 + 1 = 4.41$

$\Rightarrow x_1 = 3.41$ (1 pt)

We take $\delta = \min \{3 - 2.61, 3.41 - 3\} < 0.39$ (or any smaller test number) (1 pt)
(6 points) Find an equation for the tangent to the curve of \( g(x) = \frac{3}{\sqrt{2x+7}} \) at the point \((1, 1)\). (You must use limits)

\[
m = \text{Slope} = \lim_{h \to 0} \frac{1}{h} \left[ \frac{3}{\sqrt{2(1+h)+7}} - \frac{3}{\sqrt{9}} \right] \quad (1 \text{ pt})
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ \frac{3}{\sqrt{2h+9}} - 1 \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ \frac{3 - \sqrt{2h+9}}{\sqrt{2h+9}} \right]
\]

\[
= \lim_{h \to 0} \frac{9 - (2h+9)}{\sqrt{2h+9} \cdot (3 + \sqrt{2h+9})}
\]

\[
= \lim_{h \to 0} \frac{-2h}{\sqrt{2h+9} \cdot (3 + \sqrt{2h+9})}
\]

\[
= \frac{-2}{(3)(6)} = -\frac{1}{9} \quad (2 \text{ pt})
\]

The equation is given by

\[
y - y_1 = m(x - x_1) \quad (1 \text{ pt})
\]

\[
\Rightarrow y - 1 = -\frac{1}{9}(x - 1) \quad \text{or} \quad (1 \text{ pt})
\]

\[
y = -\frac{1}{9}x + \frac{10}{9}
\]
b) (4 points) Use limits to find all vertical asymptotes of the graph of 
\[ f(x) = \frac{|x - 1|}{x(x^2 - 1)}. \]

The zeros of the denominator are \( x = 0, x = \pm 1 \).

\( x = 0 \):
\[ \lim_{x \to 0^+} \frac{|x - 1|}{x(x - 1)(x + 1)} = \lim_{x \to 0^+} \frac{-(x-1)}{x(x-1)(x+1)} = +\infty \]

\( x = -1 \):
\[ \lim_{x \to -1^\pm} \frac{-|x - 1|}{x(x - 1)(x + 1)} = \pm \infty \]

\( x = 1 \):
\[ \lim_{x \to 1^+} \frac{|x - 1|}{x(x - 1)(x + 1)} = \lim_{x \to 1^-} \frac{|x - 1|}{x(x - 1)(x + 1)} = \frac{1}{2} \]

\[ \lim_{x \to 1^+} \frac{-(x-1)}{x(x-1)(x+1)} = -\frac{1}{2} \]

\[ \therefore \text{V.A are} \]
\[ x = 0 \text{ and } x = -1 \text{ only} \]
\[ \varepsilon \times 5(b) \]

\[ 0 < |x - 3| < \delta \quad \Rightarrow \quad |f(x) - 4| < 0.2 \]

\[ -5 < x - 3 < 5 \]

\[ -5 + 3 < x < 5 + 3 \]

\[ -2 < x < 8 \]

\[ -0.2 < 2\sqrt{x+1} - 4 < 0.2 \]

\[ 3.8 < 2\sqrt{x+1} < 4.2 \]

\[ 1.9 < \sqrt{x+1} < 2.1 \]

\[ 3.61 < x+1 < 4.41 \]

\[ 2.61 < x < 3.41 \]

\[ 5 + 3 = 3.41 \]

\[ 8 = 0.39 \]

\[ 8 = 0.41 \]