1. (10 points) If $A < B$ and $y = 3e^{Ax} + 2e^{Bx}$ is a solution of the differential equation $y'' + 2y' - 3y = 0$. Find $A$ and $B$.

$$y' = 3Ae^{Ax} + 2Be^{Bx}$$
$$y'' = 3Ae^{Ax} + 2B^2e^{Bx}$$

$$y'' + 2y' - 3y = (3A^2 + 6A - 9)e^{Ax} + (2B^2 + 4B - 6)e^{Bx} = 0$$

So $3A^2 + 6A - 9 = 0$, $2B^2 + 4B - 6 = 0$

i.e. $(A - 1)(A + 3) = 0$, $(B - 1)(B + 3) = 0$.

Since $A < B$ we get $A = -3$ and $B = 1$
2. (10 points) Solve the initial value problem

\[ \frac{dy}{dx} + xy = 2xe^{x^2/2}, \ y(0) = 2. \]

The equation is linear with integrating factor 
\[ \mu = e^\int x \, dx = e^{x^2/2} \]

Hence
\[ \frac{d}{dx} (\mu y) = \mu \left( 2xe^{x^2/2} \right) \]

i.e.
\[ \mu y = \int 2xe^{x^2/2} \, dx = e^{x^2} + C \]

So
\[ y = e^{-x^2/2} \left( e^{x^2} + C \right) \]

\( y(0) = 2 \) gives \( 2 = 1 + C \), i.e. \( C = 1 \).

hence IVP has solution
\[ y = e^{-x^2/2} \left( e^{x^2} + 1 \right) \]

(or \( y = e^{x^2/2} + e^{-x^2/2} \))
3. (10 points) Find the general solution of the differential equation

\[ x \frac{dy}{dx} + 6y = 3xy^{4/3}. \]

DE is \( \frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3} \)

Bernoulli with \( r = \frac{4}{3} \)

Put \( u = y^{-1/3} \), then \( \frac{du}{dx} = \frac{1}{3} y^{-4/3} \frac{dy}{dx} \)

DE becomes

\[ \frac{du}{dx} - \frac{2}{x} u = -1 \]

linear 1st order

Integrating factor \( p = e^{-\int \frac{2}{x} \, dx} = x \)

Hence \( \frac{d}{dx} (pu) = -p \)

i.e. \( pu = -\int p \, dx \) and we get

\[ u = x^2 \left( \frac{1}{x^2} + C \right) \]

i.e. \( y = \left( x + C \cdot x^3 \right)^{-3} \)
4. (10 points) Use the method of Gauss-Jordan elimination to solve the system

\[ x_1 + 3x_2 + 3x_3 = 13 \]
\[ 2x_1 + 5x_2 + 4x_3 = 23 \]
\[ 2x_1 + 7x_2 + 8x_3 = 29 \]

Augmented matrix

\[
\begin{bmatrix}
1 & 3 & 3 & 13 \\
2 & 5 & 4 & 23 \\
2 & 4 & 8 & 29 \\
\end{bmatrix}
\]

\[
\begin{align*}
R_2 & \rightarrow 2R_1 \\
R_3 & \rightarrow 2R_1 \\
R_3 & \rightarrow R_2 \\
R_3 & \rightarrow 3R_3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 3 & 3 & 13 \\
0 & -1 & -2 & -3 \\
0 & 1 & 2 & 3 \\
\end{bmatrix}
\]

RREF

So \( x_3 = t \), \( x_2 = 3 - 2t \), \( x_1 = 4 + 3t \)
5. (10 points) Solve the initial value problem

\[
\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}, \quad y(0) = 1
\]

\[
\frac{dy}{dx} = \frac{(x-1)(y+3)}{(x+4)(y-2)}
\]

Separable

\[
\frac{y-2}{y+3} \, dy = \frac{x-1}{x+4} \, dx
\]

\[
y+3-5 \cdot \frac{dy}{y+2} = \frac{x+4-5}{x+4} \, dx
\]

\[
\int \left(1- \frac{5}{y+3}\right) \, dy = \int \left(1- \frac{5}{x+4}\right) \, dx
\]

\[
y - 5 \ln |y+3| = x - 5 \ln |x+4| + C
\]

\[
y(0)=1 \text{ gives } -5 \ln 4 = -5 \ln 4 + C \text{ i.e. } C = 1
\]

So IVP has solution

\[
y - 5 \ln |y+3| = x - 5 \ln |x+4| + 1
\]
6. (10 points) Solve the initial value problem

\( \frac{dy}{dx} = \frac{x^2}{(1+x^2)^3}, \quad y(0) = 1. \)

Separable DE

\[
\frac{dy}{dx} = \frac{x^2}{(1+x^2)^3}
\]

\[
\int dy = \int \frac{x^2 dx}{(1+x^2)^3}
\]

Put \( 1+x^2 = v \) so \( 2x^2 dx = dv \)

\[
y = \frac{1}{3} \int \frac{dv}{v^3} = -\frac{1}{6v^2} + C
\]

i.e. \( y = -\frac{1}{6(1+x^2)} + C \)

\( y(0) = 1 \) given \( 1 = -\frac{1}{6} + C \) i.e. \( C = \frac{7}{6} \)

IVP has solution

\[
y = -\frac{1}{6(1+x^2)^3} + \frac{7}{6}
\]
7. (10 points) Verify that the differential equation

\[(y^2 + \cos x - 1)\, dx + (2xy + \cos y + 1)\, dy = 0\]

is exact and then find its general solution.

Put \( M = y^2 + \cos x - 1 \), \( N = 2xy + \cos y + 1 \),

then \( M_y = 2y \) and \( N_x = 2y \). Since \( M_y = N_x \),

the DE is exact.

Hence there is a function \( F \) such that

\[ F_x = M \text{ and } F_y = N. \]

We have \( F_x = y^2 + \cos x - 1 \)

so \( F = xy^2 + \sin x - x + g(y) \)

\[ F_y = 2xy + g'(y) = 2xy + \cos y + 1 \]

So \( g(y) = \int (\cos y + 1)\, dy = \sin y + y + C \).

Solution is \( F(xy) = \text{constant i.e.} \)

\[ xy^2 + \sin x - x + \sin y + y + C = 0 \]

(or \( xy^2 + \sin x + \sin y - x + y = C \))
8. (10 points) Determine for what values of \( k \) the system

\[
\begin{align*}
x + 2y + z &= 2 \\
2x - y - 3z &= 5 \\
4x + 3y - z &= k
\end{align*}
\]

has

a) a unique solution  b) no solution  c) infinitely many solution.

Augmented matrix

\[
\begin{bmatrix}
1 & 2 & 1 & 2 \\
2 & -1 & -3 & 5 \\
4 & 3 & -1 & k
\end{bmatrix}
\]

\[
\begin{align*}
R_2 - 2R_1 & \rightarrow \begin{bmatrix}
1 & 2 & 1 & 2 \\
0 & -5 & -5 & 4 \\
4 & 3 & -1 & k
\end{bmatrix} \\
R_3 - 4R_1 & \rightarrow \begin{bmatrix}
1 & 2 & 1 & 2 \\
0 & -5 & -5 & 4 \\
0 & -5 & -5 & k-8
\end{bmatrix}
\end{align*}
\]

If \( k-8 = 0 \) (i.e. \( k=9 \)), the system has infinitely many solutions because there will be one free variable (only 2 leading variables).

If \( k-9 \neq 0 \) (i.e. \( k \neq 9 \)), the system is inconsistent.

There are no values of \( k \) for which the system has a unique solution.