## Math421: Introduction to Topology

Name : ........................................... ......... .........

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Exercise 1. Let $\mathcal{U}$ be the usual topology on $\mathbb{R}$. Consider the two topologies on $\mathbb{R}$ defined by:

$$\mathcal{LL} = \{ V \subseteq \mathbb{R} : \text{if } x \in V, \text{ then there exist } a, b \in \mathbb{R} \text{ such that } x \in [a, b] \subseteq V \}$$

and

$$\mathcal{UL} = \{ V \subseteq \mathbb{R} : \text{if } x \in V, \text{ then there exist } a, b \in V \text{ such that } x \in ]a, b[ \subseteq V \}.$$  

The topology $\mathcal{LL}$ (resp. $\mathcal{UL}$) is called the lower limit topology (resp., upper limit topology) on $\mathbb{R}$.

(1) Show that $B_1 = \{ [a, b] : a, b \in \mathbb{R} \}$ is a basis of $\mathcal{LL}$ and $B_2 = \{ ]a, b[ : a, b \in \mathbb{R} \}$ is a basis of $\mathcal{UL}$.

(2) Show that:

- $\mathcal{U} \leq \mathcal{LL}$ and $\mathcal{U} \neq \mathcal{LL}$
- $\mathcal{U} \leq \mathcal{UL}$, and $\mathcal{U} \neq \mathcal{UL}$
- $\mathcal{LL} \neq \mathcal{UL}$ and $\mathcal{UL} \neq \mathcal{LL}$. 
(3) Show that the function
\[ f : (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{U}) \]
\[ x \mapsto -x \]

is a homeomorphism.
Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} 
    x + 1 & \text{if } x > 1 \\
    x & \text{if } x \leq 1
\end{cases}$$

(4) Is $g : (\mathbb{R}, \mathcal{U}) \rightarrow (\mathbb{R}, \mathcal{U})$ continuous?

(5) Is $g : (\mathbb{R}, \mathcal{L}) \rightarrow (\mathbb{R}, \mathcal{U})$ continuous?
(6) Is \[ g : (\mathbb{R}, \mathcal{U}) \rightarrow (\mathbb{R}, \mathcal{U}) \] continuous?
Exercise 2. Let $X$ be a set and

$$\mathcal{T}_{cc} = \{ U \subseteq X : U = \emptyset \text{ or } X - U \text{ is a countable set} \}.$$ 

(a) Show that $\mathcal{T}_{cc}$ is a topology on $X$. 
(b) Show that \((X, T_{ce})\) is discrete if and only if \(X\) is countable.
Exercise 3. Let $X$ be a nonempty set and $d \in X$. Consider
\[ \mathcal{T} = \{ U \subseteq X : U = X \text{ or } d \notin U \}. \]

(a) Show that $\mathcal{T}$ is a topology on $X$.

(b) Show that for each $A \subseteq X$, we have $\overline{A} = A \cup \{d\}$.
(c) Suppose that $d \in A$; then find $\text{int}(A)$. 
(d) Suppose that \( X = \{a, b, c, d\} \). List all the open sets and closed sets of \((X, \mathcal{T})\).