Math 421: Introduction to Topology

Name : ........................................... ....... ...........

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Exercise 1. Let \((X, T)\) be a topological space.

(a) Show that \(X\) is Hausdorff if and only the diagonal

\[ \Delta = \{(x, x) : x \in X\} \]

is closed in \(X \times X\).
(b) Let $Y$ be a Hausdorff space. Show that the graph $G(f) := \{(x, f(x)) : x \in X\}$ of a continuous function $f : X \to Y$ is closed in the product space $X \times Y$. 
Exercise 2. Let $\mathcal{C} = \{(x, e^x) : x \in \mathbb{R} \}$. 

(a) Show that $\mathcal{C}$ is a closed set of $\mathbb{R}^2$. 
(b) Show that $\pi_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ (the second projection) is not a closed map.
Exercise 3. Let $X, Y$ be two topological spaces and $A \subseteq X, B \subseteq Y$.

(a) Show that
\[ \text{int}(A \times B) = \text{int}(A) \times \text{int}(B). \]

(b) Show that
\[ \overline{A \times B} = \overline{A} \times \overline{B}. \]
(c) Show that

\[ \text{Fr}(A \times B) = [\text{Fr}(A) \times \overline{B}] \cup [\overline{A} \times \text{Fr}(B)]. \]

(d) Let \( E = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \} \). Find the frontier of \( E \) in the topological space \((\mathbb{R}^2, \mathcal{U})\).
(e) Show that $A \times B$ is dense in $X \times Y$ if and only if $A$ is dense in $X$ and $B$ is dense in $Y$. 
Exercise 4. A topological space \((X, \mathcal{T})\) is said to be totally disconnected if no subspace of \(X\) of cardinality greater than or equal to 2 is connected.

(a) Show that \((\mathbb{R}, \mathcal{L})\) is totally disconnected, where \(\mathcal{L}\) is the lower limit topology on \(\mathbb{R}\).

(b) Let \((X, \mathcal{T})\) be a \(T_0\)-space such that each \(p \in X\) has a basis of clopen neighborhoods. Show that \(X\) is totally disconnected.
(c) Let \( \{ (X_i, \mathcal{T}_i) : i \in I \} \) be a family of topological spaces and \( X = \prod_{i \in I} X_i \) be the product space.

Show that \( X \) is totally disconnected if and only if \( X_i \) is totally disconnected, for each \( i \in I \).
Exercise 5. Let $\mathcal{CF}$ be the co-finite topology on $\mathbb{R}$.

(a) Show that $(\mathbb{R}, \mathcal{CF})$ is path-connected.

(b) Is there a continuous function from $(\mathbb{R}, \mathcal{CF})$ onto $(\mathbb{R}, \mathcal{LL})$?