Problem 1. Prove that if $\phi$ is a solution of the integral equation

$$y(t) = e^{it} + \alpha \int_t^\infty \sin(t - s) \frac{y(s)}{s^2} \, ds$$

(assuming the existence of the integral) then $\phi$ satisfies

$$y''(t) + \left(1 + \frac{\alpha}{t^2}\right)y(t) = 0.$$  

Define the successive approximations

$$\begin{cases} 
\phi_0(t) = 0 \\
\phi_k(t) = e^{it} + \alpha \int_t^\infty \sin(t - s) \frac{\phi_{k-1}(s)}{s^2} \, ds, & 1 \leq t < \infty
\end{cases}$$

Show by induction that $|\phi_k(t) - \phi_{k-1}(t)| \leq \frac{|\alpha|^{k-1}}{(k-1)!t^{k-1}}, 1 \leq t < \infty; k = 1, 2, \ldots$

Conclude that the $\phi_k$ are all well defined for $1 \leq t < \infty$, and the sequence $\{\phi_k\}$ converges uniformly on $[1, \infty)$ to a continuous function $\phi$.

Show that $\phi$ satisfies (1) and the estimate $|\phi(t)| \leq e^{|\alpha|}$ for $t \in [1, \infty)$.

Problem 2. Show that the solution $y = 0$ of the system $y' = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix} y$ is unstable even though the eigenvalues of the coefficient matrix are $\lambda_1(t) = \lambda_2(t) = -1$.

Problem 3. Consider the system $x' = y + x \left(1 - x^2 - y^2\right), \quad y' = -x + y \left(1 - x^2 - y^2\right)$.

Show that the linear system and the nonlinear system both have the same periodic solution. Find all other solutions and discuss their behavior with respect to the orbit of the periodic solution.

Problem 4. Consider the nonlinear oscillator with linear damping $x'' + \mu x' + ax^2 + bx^3 = 0$, with $a, b$ constants and $\mu > 0$. Find the critical points of the system and study their stability properties.

(Hint. Consider the energy: $V(x, x') = \frac{1}{2} (x')^2 + \frac{1}{2} x^2 + \frac{1}{3} ax^3 + \frac{1}{4} bx^4$. )