King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 513 Final Exam
The First Semester of 2014-2015 (141)

Time Allowed: 180 Minutes

Name: ___________________________ ID#: ______________
Section/Instructor: ___________________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

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Q:1 (25 points) Use Laplace transform method to solve the wave equation

\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = xe^{-t}, \quad 0 < x < \infty, \quad t > 0 \]

with initial conditions \( u(x, 0) = 1, u_t(x, 0) = 0, \quad 0 < x < \infty \)

and the boundary conditions

\[ u(0, t) = \cos(t), \quad \lim_{x \to \infty} |u(x, t)| = x^n, \quad n \text{ finite, } t > 0 \]
Q:2 (25 points) Solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0$$

subject to the following initial and non-homogeneous boundary conditions

$$u(0, t) = 2, \quad u_x(\pi, t) = 0, \quad t > 0, \quad u(x, 0) = 4, \quad 0 < x < \pi$$
Q:3 (25 points) Solve the Laplace equation by separation of variables

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < \pi \]

\[ u(0, y) = y, \quad u_y(x, y) |_{y=\pi} = 0 \]

\[ u_x(x, y) |_{x=1} = 0, \quad u_y(x, y) |_{y=0} = 0. \]
Q:4 (25 points) Solve

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 1, t > 0 \]

subject to following boundary conditions

\[ u(r, 0) = 1 - r^2, \quad \frac{\partial u}{\partial t} \big|_{t=0} = 0, \quad u(1, t) = 0. \]
Q:5 (25 points) Find the steady-state temperature in the sphere of radius $C$ by solving

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \quad 0 < r < C, \quad 0 < \theta < \pi$$

$$u(C, \theta) = 1 - \cos(2\theta), \quad 0 < \theta < \pi.$$ 

(Hint $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1)$).
Q:6a (10 points) Given that \( \mathcal{L}^{-1}\{\frac{s}{(s^2+1)^2}\} = \frac{1}{2} t \sin(t) \), find \( \mathcal{L}^{-1}\{\frac{1}{(s^2+1)^2}\} \).

Q:6b (10 points) Show that \( J_2(x) = J_0''(x) - \frac{J_0(x)}{x} \).
Q:7 (25 points) Use the matrix exponential to find the general solution of the following system of first-order linear ordinary differential equations
\[
\begin{align*}
x' &= x + y + 2z + t \\
y' &= -x + 3y + 4z + 1 \\
z' &= 2z + e^t.
\end{align*}
\]
(Hint: Set of fundamental solutions is \( S = \{ e^{2t}, \ te^{2t}, \ t^2 e^{2t} \} \))
(Do not evaluate the integral)