Exercise 1
Suppose that $f$ is analytic for $|z| < 2$ and $\alpha$ is a complex constant. Evaluate

$$I = \int_{|z|=1} (\text{Re } z + \alpha) \frac{f(z)}{z} \, dz$$
Exercise 2

Let $f$ be an entire function and $n \in \mathbb{N}, n \geq 1$.

(i) Find $I_f = \int_{|z|=2} \frac{f(z)}{(z - 1)^{n+2}} \, dz$.

(ii) What is the value of $I_f$ if $f$ is a polynomial of degree $n$?
Exercise 3

True or false (if true, give a short explanation, if false, give a counterexample)

(i) If $f$ is an entire function and bounded on a half-plane, then $f$ is constant.

(ii) If $f$ is analytic and bounded on $|z| > 1$, then $f$ is constant.

(iii) If $f$ is entire and bounded on $|z| > 1$, then $f$ is constant.

(iv) If $f$ is analytic in the punctured complex plane $\mathbb{C} \setminus \{0\}$ such that $f(1/n) = 0$, for all $n \geq 1$. Then $f = 0$.

(v) Suppose $f$ is analytic in the annulus $1 \leq |z| \leq R$, $|f(z)| \leq R^n$ for $|z| = R$ and $|f(z)| \leq 1$ on $|z| = 1$. Then $|f(z)| \leq |z|^n$ in the annulus.
Exercise 4

In each case, exhibit a nonconstant $f$ having the desired properties or explain why no such function exists:

(i) $f$ is analytic in $|z| < 1$ with $f\left(\frac{1}{n}\right) = \frac{1}{n^2 + 1}$ for $n \in \mathbb{N}$.

(ii) $f$ is analytic in $|z| < 1$ with $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$, $n \geq 1$.

(iii) $f$ is analytic in $|z| < 1$ with $f\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}$, $n \geq 1$.

(iv) $f$ is analytic in $\mathbb{C} \setminus \{0\}$ with $f'(z) = \frac{\cos z}{z}$.
Exercise 5

Prove the following sharper version of the Schwarz’s lemma: If \( f : \Delta \to \Delta \) is analytic with \( f(0) = f'(0) = \ldots f^{(n-1)}(0) = 0, n \in \mathbb{N}, n \geq 1 \), then

\[
|f(z)| \leq |z|^n \quad \text{for all } z \in \Delta \text{ and } |f^{(n)}(0)| \leq n!
\]

Moreover, \( f(z) = az^n \) for some \( a, |a| = 1 \) if and only if either \( |f^{(n)}(0)| = n! \) or \( |f(c)| = |c|^n \) for some \( c \in \Delta \setminus \{0\} \).
Exercise 6
Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ be the upper half-plane.

1. Show that $\phi : \mathbb{H} \to \Delta, z \mapsto \frac{z - i}{z + i}$ is 1-1 analytic mapping and find $\phi^{-1}$.

2. Let $f : \Delta \to \mathbb{H}$ be analytic, with $f(0) = i$. Show that
   
   (a) $\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}$
   
   (b) $|f'(0)| \leq 2$.

3. Find all analytic functions $f : \Delta \to \mathbb{H}$, such that $f(0) = i$ and $|f''(0)| = 2$. 