King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH 535 [Functional Analysis I]
Semester (141)

Exam II: December 01, 2014 Time allowed: 2hrs:

(Q1) (a) Let X be a complex vector space and p a real-valued functional on X such that

$$p(x+y) \leq p(x) + p(y)$$

and $p(\alpha x) = |\alpha| p(x)$ where $x, y \in X$ and α is a scalar.

If f is a linear functional on a subspace Z of X satisfying $|f(x)| \leq p(x)$ for all $x \in Z$, then prove that f has a linear extension $\tilde{f}$ from Z to X such that $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

(b) For every x in a normed space X, show that

$$\| x \| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\| f \|}.$$

(Q2) (a) Let $\{T_n\}$ be a sequence of bounded linear transformations from a Banach space X into a normed space Y such that $\| T_n x \|$ is bounded for every $x \in X$. Then the sequence $\{T_n\}$ is bounded. Show by means of an example that this principle is not valid if X is a normed space.

(b) Prove that a closed linear mapping of a Banach space X into a Banach space Y is continuous **(State needed results–do not prove them)**.

(Q3) (a) Use open mapping theorem to show that a one-to-one continuous linear map of a Banach space X onto a Banach space Y over the same field is a linear homeo morphism.

(b) Verify that for x, y in an inner product space $(X, < \cdot >)$,
$\sqrt{< x+y,\ x+y >} \leq \sqrt{< x, x >} + \sqrt{< y, y >}$.
Under what conditions the equality in it takes place?

(Q4) (a) Show that for x, y, z in an inner product space X,

$$\| z - x \|^2 + \| z - y \|^2 = \frac{1}{2} \| x - y \|^2 + 2 \| z - \frac{x+y}{2} \|^2 .$$

(b) Explain why the space $C[a, b]$ under its usual form is not an inner product space.